

Intertwiners of Representations of Quantum Affine Algebras and Yangians

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The Intertwiner

Consider the quantum affine algebra $U_q\widetilde{\mathfrak{g}}$, of level 0, associated to a simple finite dimensional Lie algebra $\mathfrak{g}$.

Let V be a finite-dimensional irreducible representation of $U_q\widetilde{\mathfrak{g}}$.

For generic z , there is a unique (up to a constant) module isomorphism

$$\check{R}(z) : V(z) \otimes V \rightarrow V \otimes V(z)$$

where $V(z)$ is the module obtained by the shift of spectral parameter.

Goal

To give explicit expressions for $\check{R}(z)$ for the first fundamental modules of all quantum affine algebras.

[ZF80], [KRS81], [IK81], [KS82], [B85], [O86], [J86], [J89], [M90], [K90],
[DGZ94], [ES02], [Ma14], [MRR16], [GRW18], [JLM18], [JLM20], [Ju20],
[DF24]

Methods

Consider the tensor product decomposition of $U_q\mathfrak{g}$ -modules

$$V \otimes V \cong m_1 L_{\lambda_1} \oplus \cdots \oplus m_n L_{\lambda_n}.$$

Since $\check{R}(z)$ is a $U_q\mathfrak{g}$ -module homomorphism, we must have

$$\check{R}(z) = \sum_{k=1}^n f_k(z) P_{\lambda_k}$$

where P_{λ_k} is the projector onto the $U_q\mathfrak{g}$ -module L_{λ_k} .

After a choice of a basis of singular vectors, $f_k(z)$ become $m_k \times m_k$ matrices whose entries are rational functions of z .

Case of Trivial Multiplicities

In the case of trivial multiplicities, that is, when $m_k = 1$ for every $1 \leq k \leq n$, we have a combinatorial algorithm using q -characters, which determines the functions $f_k(z)$ completely.

The E_8 R -Matrix

Consider the first fundamental $U_q E_8^{(1)}$ -module $\tilde{L}_1(a)$. As $U_q E_8$ -modules, we have

$$(\tilde{L}_1(a))^{\otimes 2} \cong \left(\underbrace{L_{\omega_1}}_{248} \oplus \underbrace{L_{\omega_0}}_1 \right)^{\otimes 2} \cong \underbrace{L_{2\omega_1}}_{27000} \oplus \underbrace{L_{\omega_2}}_{30380} \oplus \underbrace{L_{\omega_7}}_{3875} \oplus 3 \underbrace{L_{\omega_1}}_{248} \oplus 2 \underbrace{L_{\omega_0}}_1 .$$

The corresponding R -matrix is then given by

$$\check{R}(z) = P_{2\omega_1} + f_1(z)P_{\omega_2} + f_2(z)P_{\omega_7} + M_1(z) \otimes P_{\omega_1} + M_2(z) \otimes P_{\omega_0} ,$$

where $f_i(z) \in \mathbb{C}(z)$, $M_1(z)$ is a 3×3 matrix, $M_2(z)$ is a 2×2 matrix, both having entries in $\mathbb{C}(z)$.

Zeros and Poles of the R -matrix

The zeros of the R -matrix $\check{R}(z)$, the corresponding submodules and quotient modules are given by

Zeros	Quotient Modules	Submodules
q^2	$1_a 1_{aq^2} \cong L_{2\omega_1} \oplus L_{\omega_1} \oplus L_{\omega_0}$	$2_{aq} \cong L_{\omega_2} \oplus L_{\omega_7} \oplus 2L_{\omega_1} \oplus L_{\omega_0}$
q^{12}	$1_a 1_{aq^{12}} \cong L_{2\omega_1} \oplus L_{\omega_2} \oplus 2L_{\omega_1} \oplus L_{\omega_0}$	$7_{aq^6} \cong L_{\omega_7} \oplus L_{\omega_1} \oplus L_{\omega_0}$
q^{20}	$1_a 1_{aq^{20}} \cong L_{2\omega_1} \oplus L_{\omega_2} \oplus L_{\omega_7} \oplus 2L_{\omega_1} \oplus L_{\omega_0}$	$1_{aq^{10}} \cong L_{\omega_1} \oplus L_{\omega_0}$
q^{30}	$1_a 1_{aq^{30}} \cong L_{2\omega_1} \oplus L_{\omega_2} \oplus L_{\omega_7} \oplus 3L_{\omega_1} \oplus L_{\omega_0}$	$1 \cong L_{\omega_0}$

For every zero at q^k we have a pole at q^{-k} .

Assuming the poles and zeros are simple, this determines the rational functions, $f_1(z)$ and $f_2(z)$ completely. We get

$$f_1(z) = -q^2 \frac{1 - q^{-2}z}{1 - q^2z}, \quad f_2(z) = q^{14} \frac{(1 - q^{-2}z)(1 - q^{-12}z)}{(1 - q^2z)(1 - q^{12}z)}.$$

Properties of $M_1(z)$, $M_2(z)$

The following properties determine $M_1(z)$ and $M_2(z)$ completely up to conjugation by a diagonal matrix.

- $M_i(z; q) = M_i(z^{-1}; q^{-1})$, $i = 1, 2$.
- $M_1(z)$ commutes with $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.
- $M_1(q^2)$ is a rank 1 matrix while $M_1(q^{12})$ and $M_1(q^{20})$ are rank 2.
- $M_2(q^2)$, $M_2(q^{12})$, $M_2(q^{20})$, $M_2(q^{30})$ are all rank 1 matrices.
- At $z = 1$, both $M_1(z)$ and $M_2(z)$ are identities.
- At $z = 0$ and $z = \infty$, both $M_1(z)$ and $M_2(z)$ can be computed using the action of Casimir element.
- $M_i(z)M_i(z^{-1}) = \text{Id}$, $i = 1, 2$.

$$M_1 = \frac{1}{c_1} \begin{bmatrix} m_{11}(z) & \beta_q z(z-1) & \beta_q z(z-1) \\ \gamma_q z(z-1) & a_q z(q^{-15} + q^{15}z) & (z-1)(q^{-15} - b_q z + q^{15}z^2) \\ \gamma_q z(z-1) & (z-1)(q^{-15} - b_q z + q^{15}z^2) & a_q z(q^{-15} + q^{15}z) \end{bmatrix},$$

$$c_1 = (qz - q^{-1})(q^6 z - q^{-6})(q^{10} z - q^{-10}),$$

$$m_{11}(z) = q^{15} + q^6 \alpha_q z - q^{-6} \alpha_{q^{-1}} z^2 - q^{-15} z^3.$$

$$M_2 = \frac{1}{c_2} \begin{bmatrix} n_{11}(z) & \eta_q z(z^2 - 1) \\ \rho_q z(z^2 - 1) & z^4 n_{11}(z^{-1}) \end{bmatrix},$$

$$c_2 = (qz - q^{-1})(q^6 z - q^{-6})(q^{10} z - q^{-10})(q^{15} z - q^{-15}),$$

$$n_{11}(z) = q^{30} - q^{15} \zeta_q z + \xi_q z^2 - q^{-15} \zeta_q z^3 + q^{-30} z^4.$$

The constant β_q in M_1 is arbitrary depending on a choice of normalization of the singular vector of weight ω_0 in $L_{\omega_1} \oplus L_{\omega_0}$.

To fix it for a given choice, one can use the E_0 action, which also fixes η_q (not determined from the properties above).

The constants appearing in $M_1(z)$ and $M_2(z)$ are given as follows:

$$\alpha_q = \frac{[2]_q^{i_{19}} - [2]_q^{i_{17}} - [2]_q^{i_{13}} - 2q^{15} + q^{11} + q^9 - q^{-1}}{[2]_q^8 + [2]_q^6 - [3]_q},$$

$$\beta_q = \frac{[2]_q [2]_q^{i_2} [2]_q^{i_3}}{[2]_q^8 + [2]_q^6 - [3]_q}, \quad \gamma_q = \frac{[2]_q^{16} [2]_q^{i_5} [2]_q^{i_{15}} [3]_q^{i_3}}{[2]_q^8 + [2]_q^6 - [3]_q},$$

$$a_q = \frac{[2]_q^{i_2} [2]_q^{i_3} [2]_q^{i_5}}{[2]_q^8 + [2]_q^6 - [3]_q}, \quad b_q = \frac{[2]_q^{13} + [2]_q^{11} - [2]_q^7}{[2]_q^8 + [2]_q^6 - [3]_q},$$

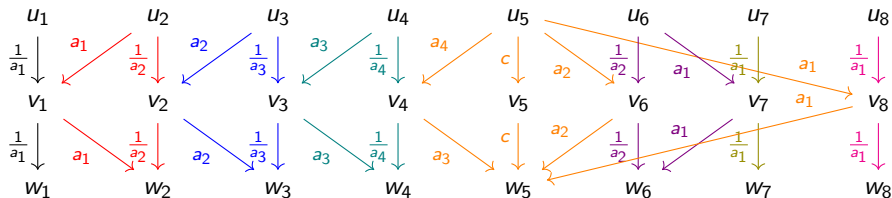
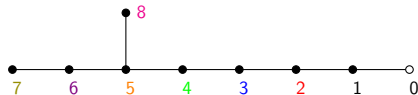
$$\zeta_q = \frac{[2]_q [2]_q^{16} [3]_q^{i_3}}{[2]_q^8 + [2]_q^6 - [3]_q}, \quad \xi_q = [2]_q^{32} - [2]_q^{30} + [2]_q^{18} + [2]_q^{10} + 2,$$

$$\eta_q = \frac{([2]_q^{i_2})^3 ([3]_q)^2}{[2]_q^8 + [2]_q^6 - [3]_q}, \quad \rho_q = \frac{[2]_q^6 [2]_q^{10} [2]_q^{12} ([2]_q^{i_5})^2 [2]_q^{i_{31}}}{[2]_q ([2]_q^8 + [2]_q^6 - [3]_q)}.$$

$$\text{Here } [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \text{ and } [n]_q^i = \frac{1}{i^{n-1}} [n]_{iq} = \frac{q^{n+(-1)^{n-1}q^{-n}}}{q + q^{-1}}.$$

Structure of L_{ω_1}

We choose a basis which is orthonormal with respect to the Shapovalov form.



$$a_i = \sqrt{\frac{[i]_q}{[i+1]_q}}, \quad c = \sqrt{\frac{[2]_q 8 + [2]_q 6 - [3]_q}{[2]_q [3]_q [5]_q}}.$$

A Library of R -Matrices !

So far we have expressions for the intertwiners corresponding to the first fundamental representations of the following:

- All Untwisted Quantum Affine Algebras and Yangians
- All Twisted Quantum Affine Algebras and Yangians

We also provide matrix unit formulas for all untwisted types except for E_8 .

In future we plan to complete the supersymmetric cases.

Thank You !

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