Purdue Mathematical Physics Seminar

Oct 21/2025

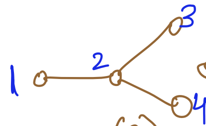
(R matrices for vector representations of Quant. Aff. Alg.)

$\mathfrak{g} \to$ simple f.d. Lie algebra of rank r

$I = \{1, \ldots, r\}$

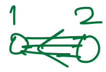
$B = DC$

(Dynkin Diagram)



$\sigma^3 = 1$

Ex. $U_q G_2$

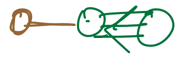


$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$U_q G_2^{(1)}$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & \end{bmatrix}\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -3 \\ -1 & -1 & 2 \end{bmatrix}$$

$U_q D_4^{(3)}$ $\quad 1 = \{1,3,4\}$, $2 = \{2\}$



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{bmatrix}$$

$U_q \mathfrak{g}$ (DJ) / $U_q \tilde{\mathfrak{g}}$ (DJ) $\quad \tilde{I}^\sigma = \underbrace{\{0, 1, \ldots, r\}}_{\text{orbits}}, \; \tilde{B}^\sigma = \tilde{D}^\sigma \tilde{C}^\sigma$

generated by $x_i^{\pm}, \; K_i, \; i \in \tilde{I}^\sigma$

$$K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i K_j = K_j K_i, \quad K_0^{a_0} K_1^{a_1} \cdots K_r^{a_r} = 1$$

$$K_i x_j^{\pm} K_i^{-1} = q^{\pm \tilde{B}_{ij}} x_j^{\pm}, \quad [x_i^{+}, x_j^{-1}] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \quad (q_i = q^{d_i})$$

+ Serre relns.

Hopf algebra

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(x_i^{\pm}) = x_i^{\pm} \otimes K_i^{1/2} + K_i^{-1/2} \otimes x_i^{\pm}, \quad i \in \tilde{I}^\sigma$$

Well-understood Rep. Theory

$V[\lambda] = \{ v \in V : K_i v = q_i^{\lambda_i} v, \; i \in I \}$ for $\lambda = (\lambda_1, \ldots, \lambda_r)$

$$V = \bigoplus_\lambda V[\lambda] \quad (\text{f.d.})$$

irred. are highest wt. cyclic modules. L_λ $\longleftrightarrow$ dominant integral weights $(\lambda_1, \ldots, \lambda_r)$, $\lambda_i \geq 0$

Shapovalov Form : unique (up to scalar) symm. bilinear form (on L_λ) $(\cdot,\cdot)$ s.t. $(X_i^+)^* = X_i^-$, $i \in I$.

irred $\Rightarrow$ non-degenerate.

Tensor Shapovalov Form on $L_\lambda \otimes L_\mu$

$$(\Delta(x_i^+))^* = \Delta(x_i^-), \quad i \in I$$

On Vector reps L_{ω_1} : $(X_i^+)^T = X_i^-$, $i \in I$

Can choose basis s.t. $\Rightarrow$

Lemma : Such a basis is orthonormal w.r.t. $(\cdot,\cdot)$ (after an appropriate choice of normalization).

$$\boxed{U_q\tilde{\mathfrak{g}} \,/\, U_q\tilde{\mathfrak{g}}^\sigma}$$

Drinfeld's new realization

The algebra $U_q\tilde{\mathfrak{g}}$ is isomorphic to the algebra with generators: $x_{i,n}^\pm$ $(i \in I, n \in \mathbb{Z})$, $\phi_{i,\pm s}^\pm$ $(i \in I, s \in \mathbb{Z}_{>0})$, $k_i^{\pm 1}$ $(i \in I)$ modulo:

$$X_i^\pm(z) = \sum_{n \in \mathbb{Z}} X_{i,n}^\pm z^n \in U_q\tilde{\mathfrak{g}}[[z,z^{-1}]], \quad \phi_i^\pm(z) = k_i^{\pm 1}\Big(1 + \sum_{s \in \mathbb{Z}_{>0}} \phi_{i,\pm s}^\pm z^{\pm s}\Big) \in U_q\tilde{\mathfrak{g}}[[z^{\pm 1}]]$$

$$k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad [\phi_i^\pm(z), \phi_j^\pm(\omega)] = [\phi_i^\pm(z), \phi_j^\mp(\omega)] = 0 \quad (1)$$

$$\phi_i^\varepsilon(z) X_j^\pm(\omega) = \underbrace{q^{B_{ij}}\left(\frac{1 - q^{\mp B_{ij}} z/\omega}{1 - q^{\pm B_{ij}} z/\omega}\right)}_{\text{Affine simple root or simple } \ell\text{-roots.}} X_j^\pm(\omega)\, \phi_i^\varepsilon(z) \quad \text{for } \varepsilon = \pm$$

$$A_{i,a}(z) = \left(q^{B_{ij}} \frac{1 - q^{-B_{ij}} z a}{1 - q^{B_{ij}} z a}\right)_{j \in I}$$

+ other relations

quadratic and cubic in $X_j^\pm(\omega)$

Twisted case σ be of order m and ζ be m^{th} primitive root of unity.

(1) + $X^{\pm}_{\sigma(i)}(z) = X^{\pm}_{i}(\zeta z)$, $\Phi^{\pm}_{\sigma(i)}(z) = \Phi^{\pm}_{i}(\zeta z)$, $K^{\pm 1}_{\sigma(i)} = K_i^{\pm 1}$, $i \in I$

$$\phi_i^{\varepsilon}(z)\, x_j^{\pm}(w) = \underbrace{\prod_{k=1}^{m} q^{\pm C_{i,\sigma^k(j)}} \left(\frac{1 - q^{\mp C_{i,\sigma^k(j)}} z/w}{1 - q^{\pm C_{i,\sigma^k(j)}} z/w} \right)}_{\text{Affine simple root or simple } \ell\text{-roots.}} x_j^{\pm}(w)\, \phi_i^{\varepsilon}(z) \quad \text{for } \varepsilon = \pm 1$$

+ other relations quadratic and cubic in $x_j^{\pm}(w)$

Hopf Algebra Automorphism $(U_q\tilde{\mathfrak{g}}^{\sigma})$

$\tau_a(x_i^{\pm}(z)) = x_i^{\pm}(az)$, $\tau_a(\phi_i^{\pm}(z)) = \phi_i^{\pm}(az)$, $i \in I^{\sigma}$

Given $U_q\tilde{\mathfrak{g}}^{\sigma}$-module V, $V(a)$ is pullback by τ_a.

$$\underbrace{V(a)}_{U_q\tilde{\mathfrak{g}}^{\sigma} \xrightarrow{\tau_a} U_q\tilde{\mathfrak{g}}^{\sigma} \xrightarrow{\pi} V}$$

ℓ-weights $\gamma = (\gamma_i^{\pm}(z))_{i \in I^{\sigma}} \in \mathbb{C}[[z^{\pm 1}]]$

generalized ℓ-wt space $V[\gamma] = \{ v \in V : (\phi_i^{\pm}(z) - \gamma_i^{\pm}(z))^{\dim(V)} v = 0,\ i \in I^{\sigma} \}$

If $V[\gamma] \neq 0$, γ is an ℓ-wt of V. $\boxed{\gamma_i^{+}(0)\, \gamma_i^{-}(\infty) = 1}$

For f.d $V = \bigoplus_{\gamma} V[\gamma]$, $V_{\lambda} = \bigoplus_{\gamma} (V[\lambda] \cap V[\gamma])$

$$\boxed{\begin{array}{l} v \in V,\ v \neq 0 \text{ is a vector of } \ell\text{-weight } \gamma \text{ if} \\ (\phi_i^{\pm}(z) - \gamma_i^{\pm}(z))\, v = 0,\ i \in I^{\sigma}. \end{array}}$$

Highest ℓ-weight rep

ℓ-singular vector : $X_i^+(z)v = 0$, $i \in I^\sigma$.

$\hookrightarrow V = U_q \mathfrak{J}^\sigma v$ for some ℓ-singular vector v.

Classification Theorem

$\mathcal{U}$: set of I^σ-tuples $P = (P_i)_{i \in I^\sigma}$ of polynomial $P_i \in \mathbb{C}[z]$ with constant term 1.

1) Every irrep is highest ℓ-wt

2) V be irrep of highest ℓ-wt γ. Then $\exists\, P \in \mathcal{U}$ s.t.

$$\gamma_i^\pm(z) = q_i^{\deg P_i} \frac{P_i(zq_i^{-1})}{P_i(zq_i)}$$

$i \in I$

$$\boxed{\gamma_i^\pm(z) = \begin{cases} q^{m \deg(P_i)} \dfrac{P_i(z^m q^{-m})}{P_i(z^m q^m)}, & i = \sigma(i) \\[2ex] q^{\deg(P_i)} \dfrac{P_i(zq^{-1})}{P_i(zq)}, & i \neq \sigma(i) \end{cases} \quad i \in I^\sigma}$$

3) Bijection: $\mathcal{U} \leftrightarrow$ irrep. isom. class.

Fundamental rep

$L_i(a)$: $P^{(i)} = (1 - \delta_{ij} z)_{j \in I^\sigma}$

$\boxed{\text{Characters}}$

$Y_{i,a}$ be I^σ-tuple. s.t. $(Y_{i,a})_j(z) = 1$ for $j \neq i$

and $(Y_{i,a})_i(z) = q_i \dfrac{1 - q_i^{-1} za}{1 - q_i za}$

$$(Y_{i,a})_i(z) = \begin{cases} q^m \dfrac{1 - q^{-m} z^m a^m}{1 - q^m z^m a^m} & \text{if } i = \sigma(i) \\[2ex] q \dfrac{1 - q^{-1} za}{1 - qza} & \text{if } i \neq \sigma(i) \end{cases}$$

$Y_{i,a}$ — highest ℓ-wt of $L_i(a)$

$Y_{i,a} = Y_{i,\zeta a}$ if $\sigma(i) = i$

- $\mathcal{Y}$ = abelian grp of I^σ-tuples of rational functions generated by $\{Y_{i,a}^{\pm 1}\}_{i \in I^\sigma, a \in \mathbb{C}^\times}$ modulo relation
- ℓ-wts of reps $\in \mathcal{Y}$.
- $\mathcal{Y}_+$ = dominant ℓ-weights monomials $= \langle Y_{i,a} \rangle_{i \in I^\sigma, a \in \mathbb{C}^\times}$
- Dominant monomials $\longleftrightarrow$ highest ℓ-wts. of irreps.

q-character map

$$\chi_q(V) = \sum_{\gamma \in \mathcal{Y}} \dim(V[\gamma])\gamma \in \mathbb{Z}_{\geq 0}[\mathcal{Y}]$$

- $\chi_q : V \mapsto \chi_q(V)$ is an injective ring homomorphism.

- Special modules : Those with unique dominant monomials
  $\hookrightarrow$ can compute $\chi_q(V)$ using FM algorithm.

Example : Denote $Y_{i,a}$ by i_a

For $U_q G_2^{(1)}$,

$$A_{1,a}(z) = 1_{aq} 1_{aq^{-1}} 2_a^{-1}$$

$$A_{2,a}(z) = 2_{aq^3} 2_{aq^{-3}} 1_{aq^2}^{-1} 1_a^{-1} 1_{aq^{-2}}^{-1}$$

$$1_a \xrightarrow{A_{1,aq}^{-1}} 1_{aq^2}^{-1} 2_{aq} \xrightarrow{A_{2,aq^4}^{-1}} 1_{aq^4} 1_{aq^6} 2_{aq^7}^{-1} \xrightarrow{A_{1,aq^7}^{-1}} 1_{aq^8}^{-1} 1_{aq^4}$$

$$1^{-1}_{aq^{12}} \xleftarrow{A^{-1}_{1,aq^{11}}} 1_{aq^{10}}\, 2^{-1}_{aq^{11}} \xleftarrow{A^{-1}_{2,aq^{8}}} 1^{-1}_{aq^{6}}\, 1^{-1}_{aq^{8}}\, 2_{aq^{5}} \xleftarrow{A^{-1}_{1,aq^{5}}}$$

dim $=7$

Example

For $U_q D_4^{(3)}$,

$$A_{1,a} = 1_{aq}\, 1_{aq^{-1}}\, 2^{-1}_{a}$$

$$A_{2,a} = 2_{aq}\, 2_{aq^{-1}}\, 1^{-1}_{a}\, 1^{-1}_{ja}\, 1^{-1}_{j^2a}$$

$j^3=1$

$$\begin{tikzcd}
1_a \arrow[r, "A^{-1}_{1,aq}"] & 1^{-1}_{aq^2}\, 2_{aq} \arrow[r, "A^{-1}_{2,aq^2}"] & 1_{jaq^2}\, 1_{j^2aq^2}\, 2^{-1}_{aq^3} \arrow[dl, "A^{-1}_{1,j^2aq^3}"'] \arrow[dr, "A^{-1}_{1,jaq^3}"] & \\
& 1^{-1}_{j^2aq^4}\, 1_{jaq^2} \arrow[dr, "A^{-1}_{1,jaq^3}"'] & & 1^{-1}_{jaq^4}\, 1_{j^2aq^2} \arrow[dl, "A^{-1}_{1,j^2aq^3}"] \\
& & 1^{-1}_{jaq^4}\, 1^{-1}_{j^2aq^4}\, 2_{aq^3} \arrow[d, "A^{-1}_{2,aq^4}"] & \\
& 1^{-1}_{aq^6} & 1_{aq^4}\, 2^{-1}_{aq^5} \arrow[l, "A^{-1}_{1,aq^5}"'] &
\end{tikzcd}$$

dim $=8$

R-matrices

Quasitriangular structure: $\quad \mathcal{R} \in U_q\mathfrak{g}^\sigma \hat{\otimes} U_q\mathfrak{g}^\sigma$

$$\Delta^{op}(x) = \mathcal{R}\,\Delta(x)\,\mathcal{R}^{-1}, \quad \forall x \in U_q\mathfrak{g}^\sigma$$

$\Delta^{op}(x) = P \circ \Delta(x)$

- $\check{R}(z) : V(z) \otimes V \longrightarrow V \otimes V(z)$ (normalized)

  $\check{R}(z) = P \circ (\pi_V \otimes \pi_V) \circ (\tau_z \otimes 1)(\mathcal{R})$

- $\check{R}(z)\,\Delta(x) = \Delta(x)\,\check{R}(z) \quad \forall x \in U_q\mathfrak{g}^\sigma$. (unique for generic z)

by Schur's Lemma

QYBE : $(\check{R}(z)\otimes I)(I\otimes \check{R}(zw))(\check{R}(w)\otimes I)$
(braid eqn) $= (I\otimes\check{R}(w))(\check{R}(zw)\otimes I)(I\otimes\check{R}(z))$

- $\check{R}(z)$ is a rational fn of z.
- If $V=\tilde{L}_i(a)$ then $\check{R}(1)=\mathrm{Id}$.
- $\check{R}(z)\,\check{R}(z^{-1}) = \mathrm{Id}$.

$q\mapsto q^{-1}$
$K_i\mapsto K_i^{-1}$

Lemma

$(U_q\tilde{\mathfrak{g}}^\sigma,\Delta)\to(U_{q^{-1}}\tilde{\mathfrak{g}}^\sigma,\Delta^{op})$

1. $\check{R}(z;q) = P\,\check{R}(z^{-1};q^{-1})\,P$
2. $\check{R}(z)$ is self-adjoint w.r.t. the tensor Shapovalov form

Lemma (DGZ)

Let V be irrep of $U_q\tilde{\mathfrak{g}}^\sigma$ s.t. $V|_{U_q\mathfrak{g}}\cong L_\lambda$

and $L_\lambda\otimes L_\lambda \cong \bigoplus_\mu L_\mu$

Then $$\check{R}(0) = \sum_\mu (-1)^\mu\, q^{(C(\mu)-C(2\lambda))/2}\, P_\mu$$

where $(-1)^\mu$: eigenvalue of P on $q\to 1$ limit of L_μ

$C(\mu) = (\mu,\mu+2\rho)$, $(\alpha_i,\alpha_j)=B_{ij}$

Theorem (FM for $U_q\tilde{\mathfrak{g}}$) / $U_q\tilde{\mathfrak{g}}^\sigma$ diff./simpler proof

$\tilde{L}_i(a)\otimes\tilde{L}_j(b)$ is reducible $\iff$ $\check{R}^{i,j}(z)$ has a

Pole or non-trivial Kernel at $z = a/b$

In that case $\frac{a}{b} = \zeta^{l} q^{k}$, $l, k \in \mathbb{Z}$

Algorithm

$$\left(\tilde{L}_1(a)\big|_{U_q\mathfrak{g}}\right)^{\otimes 2} \cong \bigoplus_{k} M_k \otimes L_{\lambda(k)}, \quad m_k = \dim M_k$$

Then $\check{R}(z) = \sum_{k} f_k(z) P_k$ where P_k Projector onto $L_{\lambda(k)}$

and $f_k(z) \in \mathrm{End}(M_k)$

Cases of Trivial multiplicity

$f_k(z)$ are rational fns.

Theorem

The Poles and zeros of $f_k(z)$ are simple.

↳ Poles at q^{l} of mult m_l $\left(-q^{-l}\frac{1-q^{l}z}{1-q^{-l}z}\right)^{m}$

$\Rightarrow$ At $z=0$, $(-1)^{m} q^{-ml}$

Poles | zeros from q-character

- $\tilde{L}_1(za) \otimes \tilde{L}_1(a)$ is irreducible unless $\chi_q(1_{az})\chi_q(1_a)$ has dominant monomial other that $1_{az}1_a$
- Happens when $z = q^{\pm 2}, q^{\pm 8}, q^{\pm 12}$.

For example, for $z = q^{-8}$

$$\chi_q(1_{aq^{-8}})\,\chi_q(1_a) = \chi_q(1_{aq^{-8}}1_a) + \chi_q(1_{aq^{-4}})$$

Claim this is special.

Theorem

Theorem 2.17. *Let V be an irreducible $U_q\tilde{\mathfrak{g}}^\sigma$-module. Let m be an i-dominant monomial in $\chi_q(V)$ of multiplicity one for some $i \in \mathrm{I}$. Let $b \in \mathbb{C}^\times$ and $m_- = mA_{i,b}^{-1}$. Suppose*

(1) The power of $Y_{i,bq^{-1}}$ in m is not greater than the power of $Y_{i,bq}$ in m. $\rightarrow$ no pole at $z=b$

(2) $mA_{i,c} \notin \chi_q(V)$ for all $c \in \mathbb{C}^\times$.

(3) $m_-A_{j,c} \notin \chi_q(V)$ for all $j \in \mathrm{I}$, $c \in \mathbb{C}^\times$ unless $(j,c) = (i,b)$.

(4) The multiplicity of m_- in $\chi_q(V)$ is not greater than one.

Then multiplicity of m_- in $\chi_q(V)$ is zero, $m_- \notin \chi_q(V)$.

$$m = 1_a 1_{aq^{-4}} 1_{aq^{-2}} 2^{-1}_{aq^{-1}} \xrightarrow{A_{1,aq^{-1}}} 1_{aq^{-4}} = m_-$$

$$\Rightarrow 1_{aq^{-4}} \notin \chi_q(1_{aq^{-8}} 1_a)$$

Theorem [C00 | K02 for untwisted case]

Theorem 2.10. *Let $i, j \in \mathrm{I}^\sigma$. The tensor product $\tilde{L}_i(a) \otimes \tilde{L}_j(b)$ is cyclic on the tensor product of highest weight vectors if $a/b \neq \omega^l q^k$ where $l \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$.*

($l=0$ in untwisted)

$\Rightarrow \tilde{L}_1(aq^{-4})$ is a submodule in $\tilde{L}_1(aq^{-8}) \otimes \tilde{L}_1(a)$

$\Rightarrow z = q^{-8}$ is a zero of $\check{R}(z)$.

$$\tilde{L}_1(2a)\big|_{U_qG_2} \otimes \tilde{L}_1(a)\Big|_{U_qG_2} \cong L_{\omega_1} \otimes L_{\omega_1} \cong L_{2\omega_1} \oplus L_{\omega_2} \oplus L_{\omega_1} \oplus L_{\omega_0}$$

Lemma 5.10. *The poles of the R-matrix $\check{R}(z)$, the corresponding submodules and quotient modules are given by*

Poles	*Submodules*	*Quotient modules*
q^2	$\tilde{L}_{1_a 1_{aq^{-2}}} \cong L_{2\omega_1} \oplus L_{\omega_1}$	$\tilde{L}_{2_{aq^{-1}}} \cong L_{\omega_2} \oplus L_{\omega_0}$
q^8	$\tilde{L}_{1_a 1_{aq^{-8}}} \cong L_{2\omega_1} \oplus L_{\omega_2} \oplus L_{\omega_0}$	$\tilde{L}_{1_{aq^{-4}}} \cong L_{\omega_1}$
q^{12}	$\tilde{L}_{1_a 1_{aq^{-12}}} \cong L_{2\omega_1} \oplus L_{\omega_2} \oplus L_{\omega_1}$	$\tilde{L}_1 \cong L_{\omega_0}$

□

$$\check{R}(z) = P_{2\omega_1} + f_1(z) P_{\omega_2} + f_2(z) P_{\omega_1} + f_3(z) P_{\omega_0}$$

$$f_1(z) = -q^{-2}\frac{1-q^2z}{1-q^{-2}z}, \qquad f_2(z) = -q^{-8}\frac{1-q^8z}{1-q^{-8}z}, \qquad f_3(z) = q^{-14}\frac{(1-q^2z)(1-q^{12}z)}{(1-q^{-2}z)(1-q^{-12}z)}.$$

$$\boxed{\text{Cases of nontrivial multiplicity}}$$

We construct such a basis for each type by a direct computation. In fact, the basis we choose is also orthonormal with respect to the Shapovalov form, and in addition to (2.7) we have $E_j^T = F_j$, $j \in \mathrm{I}$.

Let $t : V \to V$ be a linear map such that $v_i \mapsto \bar{v}_i$. Note that $t^2 = \mathrm{Id}$.

Lemma 2.22. *Let V be the first fundamental representation of $U_q\tilde{\mathfrak{g}}$ where $\mathfrak{g}$ is not of type A or E_6. Then*

$$\check{R}^{V,V}(z) = (t \otimes t) P \check{R}^{V,V}(z) P (t \otimes t)\ . \tag{2.8}$$

Here P is the flip operator.

Corollary

$$P f_k(z) P = f_k(z)$$

Conjecture 3.4. *Suppose $V(a) \otimes V$ has a single non-trivial submodule. Then the normalized R-matrix $\check{R}^{V,V}(z)$ has at most simple pole at $z = a$.*

In general we expect that the order of the pole at $z = a$ is at most one less than the number of irreducible subfactors.

$U_q D_4^{(3)}$

$$\mathcal{I}_1(a)\big|_{U_q G_2} \cong L_{\omega_1} \oplus L_{\omega_0}$$

u_1, u_2, u_3 $\quad w_1, w_2$

$$\left(\mathcal{I}_1(a)^{\otimes 2}\right)\Big|_{U_q G_2} \cong L_{2\omega_1} \oplus L_{\omega_2} \oplus 3L_{\omega_1} \oplus 2L_{\omega_0}$$

(after choosing sing. vectors)

3×3 $\quad 2\times 2$

$$\Rightarrow \check{R}(z) = P_{2\omega_1} + f(z) P_{\omega_2} + g_1(z) P_{\omega_1} + g_2(z) P_{\omega_0}$$