Intertwiners of Representations of Quantum Affine Algebras and Yangians

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The Intertwiner

Consider the quantum affine algebra $U_q\widetilde{\mathfrak{g}}$, of level 0, associated to a simple finite dimensional Lie algebra $\mathfrak{g}$.
Let V be a finite-dimensional irreducible representation of $U_q\widetilde{\mathfrak{g}}$.
For generic z, there is a unique (up to a constant) module isomorphism

$$\check{R}(z) : V(z) \otimes V \to V \otimes V(z)$$

where $V(z)$ is the module obtained by the shift of spectral parameter.

Goal

To give explicit expressions for $\check{R}(z)$ for the first fundamental modules of all quantum affine algebras.

[ZF80], [KRS81], [IK81], [KS82], [B85], [O86], [J86], [J89], [M90], [K90], [DGZ94], [ES02], [Ma14], [MRR16], [GRW18], [JLM18], [JLM20], [Ju20], [DF24]

Methods

Consider the tensor product decomposition of $U_q\mathfrak{g}$-modules

$$V \otimes V \cong m_1 L_{\lambda_1} \oplus \cdots \oplus m_n L_{\lambda_n} \ .$$

Since $\check{R}(z)$ is a $U_q\mathfrak{g}$-module homomorphism, we must have

$$\check{R}(z) = \sum_{k=1}^{n} f_k(z) P_{\lambda_k}$$

where P_{λ_k} is the projector onto the $U_q\mathfrak{g}$-module L_{λ_k}.
After a choice of a basis of singular vectors, $f_k(z)$ become $m_k \times m_k$ matrices whose entries are rational functions of z.

Case of Trivial Multiplicities

In the case of trivial multiplicities, that is, when $m_k = 1$ for every $1 \leq k \leq n$, we have a combinatorial algorithm using q-characters, which determines the functions $f_k(z)$ completely.

The E_8 R-Matrix

Consider the first fundamental $U_qE_8^{(1)}$-module $\widetilde{L}_1(a)$. As U_qE_8-modules, we have

$$\left(\widetilde{L}_1(a)\right)^{\otimes 2} \cong \big(\underbrace{L_{\omega_1}}_{248} \oplus \underbrace{L_{\omega_0}}_{1}\big)^{\otimes 2} \cong \underbrace{L_{2\omega_1}}_{27000} \oplus \underbrace{L_{\omega_2}}_{30380} \oplus \underbrace{L_{\omega_7}}_{3875} \oplus 3\underbrace{L_{\omega_1}}_{248} \oplus 2\underbrace{L_{\omega_0}}_{1} .$$

The corresponding R-matrix is then given by

$$\check{R}(z) = P_{2\omega_1} + f_1(z)P_{\omega_2} + f_2(z)P_{\omega_7} + M_1(z) \otimes P_{\omega_1} + M_2(z) \otimes P_{\omega_0} ,$$

where $f_i(z) \in \mathbb{C}(z)$, $M_1(z)$ is a 3×3 matrix, $M_2(z)$ is a 2×2 matrix, both having entries in $\mathbb{C}(z)$.

Zeros and Poles of the R-matrix

The zeros of the R-matrix $\check{R}(z)$, the corresponding submodules and quotient modules are given by

Zeros	Quotient Modules	Submodules
q^2	$1_a 1_{aq^2} \cong L_{2\omega_1} \oplus L_{\omega_1} \oplus L_{\omega_0}$	$2_{aq} \cong L_{\omega_2} \oplus L_{\omega_7} \oplus 2L_{\omega_1} \oplus L_{\omega_0}$
q^{12}	$1_a 1_{aq^{12}} \cong L_{2\omega_1} \oplus L_{\omega_2} \oplus 2L_{\omega_1} \oplus L_{\omega_0}$	$7_{aq^6} \cong L_{\omega_7} \oplus L_{\omega_1} \oplus L_{\omega_0}$
q^{20}	$1_a 1_{aq^{20}} \cong L_{2\omega_1} \oplus L_{\omega_2} \oplus L_{\omega_7} \oplus 2L_{\omega_1} \oplus L_{\omega_0}$	$1_{aq^{10}} \cong L_{\omega_1} \oplus L_{\omega_0}$
q^{30}	$1_a 1_{aq^{30}} \cong L_{2\omega_1} \oplus L_{\omega_2} \oplus L_{\omega_7} \oplus 3L_{\omega_1} \oplus L_{\omega_0}$	$1 \cong L_{\omega_0}$

For every zero at q^k we have a pole at q^{-k}.
Assuming the poles and zeros are simple, this determines the rational functions, $f_1(z)$ and $f_2(z)$ completely. We get

$$f_1(z) = -q^2 \frac{1-q^{-2}z}{1-q^2 z}\ , \quad f_2(z) = q^{14} \frac{(1-q^{-2}z)(1-q^{-12}z)}{(1-q^2 z)(1-q^{12} z)}\ .$$

Commutation with E_0

We use commutation with E_0 to determine the matrices $M_1(z)$ and $M_2(z)$.

$$M_1 = \frac{1}{c_1}\begin{bmatrix} m_{11}(z) & \beta_q\, z(z-1) & \beta_q\, z(z-1) \\ \gamma_q\, z(z-1) & a_q\, z(q^{-15}+q^{15}z) & (z-1)(q^{-15}-b_q\, z+q^{15}z^2) \\ \gamma_q\, z(z-1) & (z-1)(q^{-15}-b_q\, z+q^{15}z^2) & a_q\, z(q^{-15}+q^{15}z) \end{bmatrix},$$

$$c_1 = (qz-q^{-1})(q^6z-q^{-6})(q^{10}z-q^{-10})\,,$$
$$m_{11}(z) = q^{15}+q^6\alpha_q\, z - q^{-6}\alpha_{q^{-1}}\, z^2 - q^{-15}\, z^3.$$

$$M_2 = \frac{1}{c_2}\begin{bmatrix} n_{11}(z) & \eta_q\, z(z^2-1) \\ \rho_q\, z(z^2-1) & z^4 n_{11}(z^{-1}) \end{bmatrix},$$
$$c_2 = (qz-q^{-1})(q^6z-q^{-6})(q^{10}z-q^{-10})(q^{15}z-q^{-15})\,,$$
$$n_{11}(z) = q^{30}-q^{15}\,\zeta_q\, z+\xi_q\, z^2-q^{-15}\,\zeta_q\, z^3+q^{-30}z^4.$$

The constants appearing in $M_1(z)$ and $M_2(z)$ are given as follows:

$$\alpha_q = \frac{[2]^i_{q^{19}} - [2]_{q^{17}} - [2]^i_{q^{13}} - 2q^{15} + q^{11} + q^9 - q^{-1}}{[2]_{q^8} + [2]_{q^6} - [3]_q} ,$$

$$a_q = \frac{[2]^i_{q^2}\,[2]^i_{q^3}\,[2]^i_{q^5}}{[2]_{q^8} + [2]_{q^6} - [3]_q} , \; b_q = \frac{[2]_{q^{13}} + [2]_{q^{11}} - [2]_{q^7}}{[2]_{q^8} + [2]_{q^6} - [3]_q} ,$$

$$\beta_q = \frac{\left([2]^i_{q^2}\right)^2 [3]_q}{[2]_{q^8} + [2]_{q^6} - [3]_q} ,$$

$$\gamma_q = \frac{[2]_{q^{16}} \left([2]^i_{q^5}\right)^2 [3]_q\,[3]^i_{q^3}}{[2]_{q^8} + [2]_{q^6} - [3]_q} \left([2]_{q^4} + [2]_{q^3} - [2]_q - 1\right)\left([2]_{q^4} - [2]_{q^3} + [2]_q - 1\right) ,$$

$$\zeta_q = \frac{[2]_q\,[2]_{q^{16}}\,[3]^i_{q^3}}{[2]_{q^8} + [2]_{q^6} - [3]_q} , \quad \xi_q = [2]_{q^{32}} - [2]_{q^{30}} + [2]_{q^{18}} + [2]_{q^{10}} + 2 ,$$

$$\eta_q = \frac{\left([2]^i_{q^2}\right)^3 \left([3]_q\right)^2}{q\left([2]_{q^8} + [2]_{q^6} - [3]_q\right)^2} , \quad \rho_q = \frac{q\,[2]_{q^6}\,[2]_{q^{10}}\,[2]_{q^{12}} \left([2]^i_{q^5}\right)^2 [2]^i_{q^{31}}}{[2]_q} .$$

Here $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$ and $[n]^i_q = \frac{1}{i^{n-1}}[n]_{iq}$.

A Library of R-Matrices !

So far we have expressions for the intertwiners corresponding to the first fundamental representations of the following:

- All Untwisted Quantum Affine Algebras and Yangians
- All Twisted Quantum Affine Algebras and Yangians

In future we plan to complete the supersymmetric cases.

Thank You !

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