- Intervals: open, closed, half-open, infinite
- Functions: independent and dependent variables
  inputs outputs
- Vertical line test
- Domain and Range
- Given f, find $f(a+1)$, $f(a^2)$, $f(x+h)$ etc.
- Composition
- Compound functions

dependent variable

$y = f(x)$

$f(x) = x^2$

independent variable (input)

1 Intervals

sub-collection

(a) Open interval (subset of real numbers)

$$(4, 5) = \{x \text{ real} : 4 < x < 5\}$$

(b) Closed interval (subset of real numbers)

$$[4, 5] = \{x \text{ real} : 4 \leq x \leq 5\}$$

(c) Half-open intervals

$$[4, 5) = \{x \text{ real} : 4 \leq x < 5\}$$

$$(4, 5] = \{x \text{ real} : 4 < x \leq 5\}$$

(d) Infinite intervals

∞ → larger than every real number

$-\infty$ → smaller than every real number

- $(-\infty, \infty) = \{x \text{ real} : -\infty < x < \infty\}$.

- $[-\infty, 4)$ → not a sub-collection of real number

  → $-\infty$ is not a real number

- $(4, \infty]$ ✗

  → ∞ is not a real number

- $[4, \infty) = \{x \text{ real} : 4 \le x < \infty\}$
- $(-\infty, 4] = \{x \text{ real} : -\infty < x \le 4\}$
- $(4, \infty)$
- $(-\infty, 4)$

2 Functions

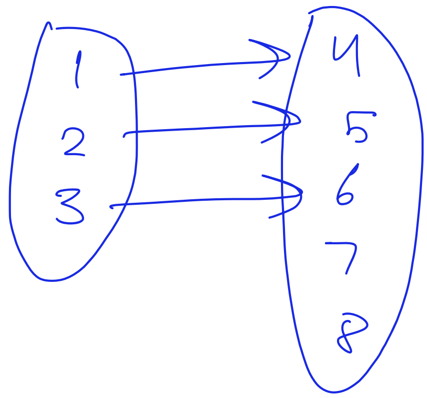
Functions → rules | assignments that relate one sub-collection of reals to another sub-collection of reals such that :—

Input → Output

1) every member of the input is assigned a unique member of the output

* functions are usually denoted by f, g, h etc..

Example $x \longmapsto x+3$



$$f(x) = x+3$$

f of x

The function here is f.

Example $f(x) = x^2$ $\quad x \longmapsto x^2$

$$f(4) = 4^2 = 16$$

Example $f(x) = x^2$

$$f(a+h) = ??$$

$$\Rightarrow f(a+h) = (a+h)^2$$
$$= (a+h)(a+h)$$
$$= a^2 + ha + ah + h^2$$
$$= a^2 + 2ah + h^2$$

$f(a-1) =$ HW exercise

4 Domain and Range

Domain — all possible real numbers for which f is applicable.

that is, no mathematical law is violated while applying f.

- **Example**

$f(x) = x+3$

Domain of f = all real numbers

$= (-\infty, \infty)$

- **Example**

$$f(x) = \frac{1}{x+3}$$

$$f(0) = \frac{1}{0+3} = \frac{1}{3} = 0.333\ldots$$

$$f(1) = \frac{1}{1+3} = \frac{1}{4} = 0.25$$

$$f(2) = \frac{1}{2+3} = \frac{1}{5} = 0.2$$

$\vdots$

$$f(-1) = \frac{1}{-1+3} = \frac{1}{2} = 0.5$$

$$f(-2) = \frac{1}{-2+3} = 1$$

$$f(-3) = \frac{1}{-3+3} = \frac{1}{0}$$

→ Cannot divide by 0 or $\frac{1}{0}$ is not a real number.

$$f(-4) = \frac{1}{-4+3} = \frac{1}{-1} = -1$$

$x+3 \neq 0$

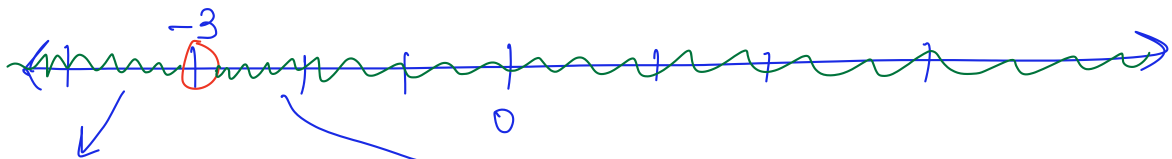
$x+3 = 0 \Rightarrow x = -3$ (this should not happen)

$x \neq -3 \Rightarrow$ Domain of f = all real number except -3

$\mathbb{R} \setminus \{-3\}$

$\mathbb{R} \setminus \{-3\}$

$(-\infty, -3) \cup (-3, \infty)$



$(-\infty, -3) \quad \cup \quad (-3, \infty)$

union

Domain of $f = \boxed{(-\infty, -3) \cup (-3, \infty)}$

Example

$f(x) = \sqrt{3-x}$

$3 - x \geq 0$

$-1(x-3) \geq 0$

n is even	n is odd
$\sqrt{x}$	$\sqrt[3]{x}$
$\sqrt[4]{x}$	$\sqrt[5]{x}$
$\sqrt[6]{x}$	$\sqrt[7]{x}$
⋮	⋮

$$\frac{\cancel{-1}(x-3)}{\cancel{-1}} \leq \frac{0}{-1}$$

$x \geq 0$

no restriction on x

$$x-3 \leq 0$$

$$x \leq 3$$

$$x-3+3 \leq 0+3$$

$$3-x+x \geq 0+x$$

$$3 \geq x \Rightarrow x \leq 3$$

$\Rightarrow$ Domain of $f = (-\infty, 3]$

$$\boxed{(3-x)(2+x) \geq 0}$$

Example

$$f(x) = \sqrt{x^2-4}$$

$$a^2-b^2=(a+b)(a-b)$$

$$x^2-4 \geq 0$$

$$x^2-2^2 \geq 0$$

$$(x+2)(x-2) \geq 0$$

$$\Downarrow$$

$ab \geq 0$

$\Downarrow$

$a \geq 0$ and $b \geq 0$

or

$a \leq 0$ and $b \leq 0$

Case 1: $x+2 \geq 0$ and $x-2 \geq 0 \Rightarrow x \geq -2$ and $x \geq 2$

$\Downarrow$

$x \geq 2$

or

Case 2: $x+2 \leq 0$ and $x-2 \leq 0 \Rightarrow x \leq -2$ and $x \leq 2$

$\Downarrow$

$x \leq -2$

$\Rightarrow x \geq 2$ or $x \leq -2$

Domain of $f = (-\infty, -2] \cup [2, \infty)$

$a < b$

- $(x-a)(x-b) \geq 0$

$$\Downarrow$$

x lies in

$(-\infty, a] \cup [b, \infty)$

- $(x-a)(x-b) > 0$

$$\Downarrow$$

$(-\infty, a) \cup (b, \infty)$

Example

$(x-2)(x-3) > 0$

$\Rightarrow x$ is in $(-\infty, 2) \cup (3, \infty)$

Example

$$f(x) = \sqrt{4 - x^2}$$

$4 - x^2 \geq 0 \Rightarrow 4 - x^2 + x^2 \geq 0 + x^2$

$2^2 - x^2 \geq 0 \quad \Rightarrow 4 \geq x^2$

$(2-x)(2+x) \geq 0 \quad \Downarrow$

$x^2 \leq 4$

$\Downarrow$

$x^2 - 4 \leq 0 \Rightarrow x^2 - 2^2 \leq 0$

$\Rightarrow (x-2)(x+2) \leq 0$

Case 1 $\quad x - 2 \leq 0$ and $x + 2 \geq 0 \Rightarrow -2 \leq x \leq 2$

or

Case 2 $\quad x - 2 \geq 0$ and $x + 2 \leq 0 \Rightarrow$ empty set

$\Rightarrow -2 \leq x \leq 2 \Rightarrow$ Domain of $f = [-2, 2]$

Let $a<b$.

- $(x-a)(x-b) \leq 0$

$\Rightarrow a \leq x \leq b$ or x lies in $[a, b]$

- $(x-a)(x-b) < 0$

$\Rightarrow a < x < b$ or x lies in (a, b)

<u>Range of f</u> : All possible values of $f(x)$ as x <u>varies</u> in the domain of f

$y = \sqrt{3-x}$

Domain of f was $(-\infty, 3]$

$x=3 \Rightarrow y = \sqrt{3-3} = 0$

$x=2 \Rightarrow y = \sqrt{3-2} = \sqrt{1} = 1$

$x=-1 \Rightarrow y = \sqrt{3-(-1)} = \sqrt{4} = 2$

as x decreases $\qquad$ y increases

y lies in $[0, \infty)$

$\Rightarrow$ Range of $f = [0, \infty)$

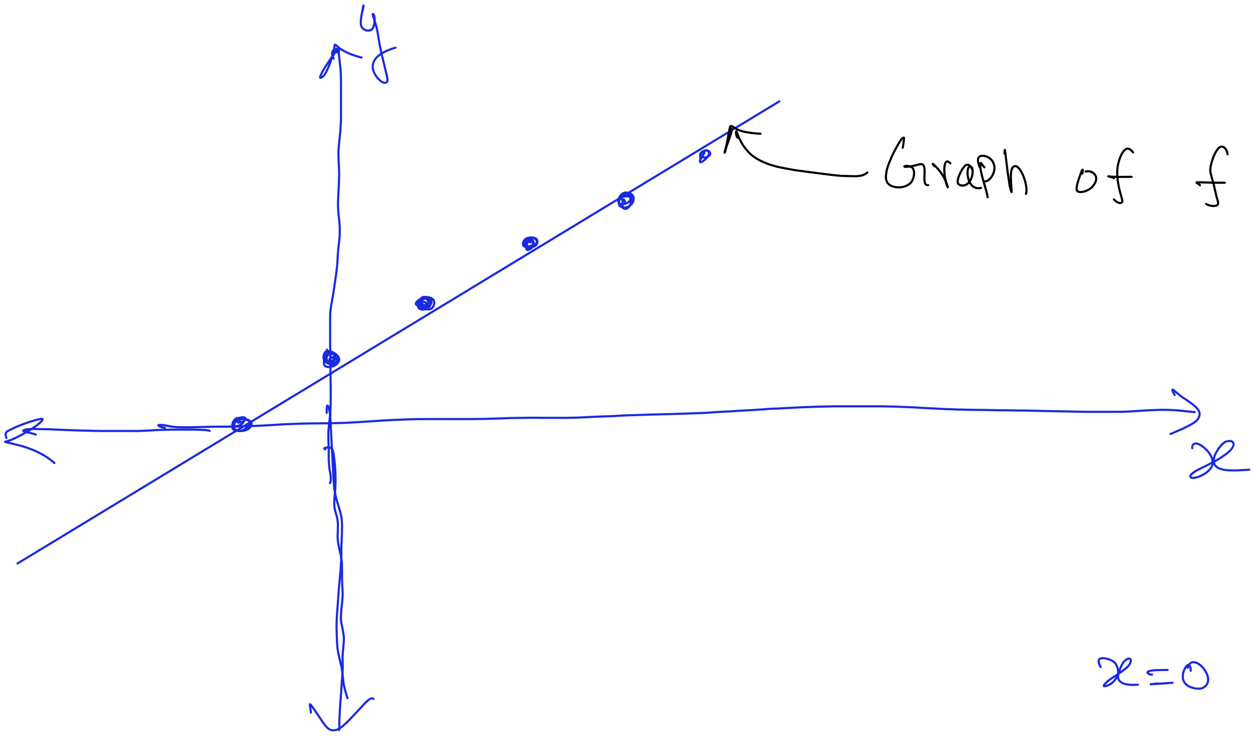
3. Graph of a function (vertical line test)

$f(x) = x+1$

$y = x+1$

$x \rightarrow$ independent variable

$y \rightarrow$ dependent variable

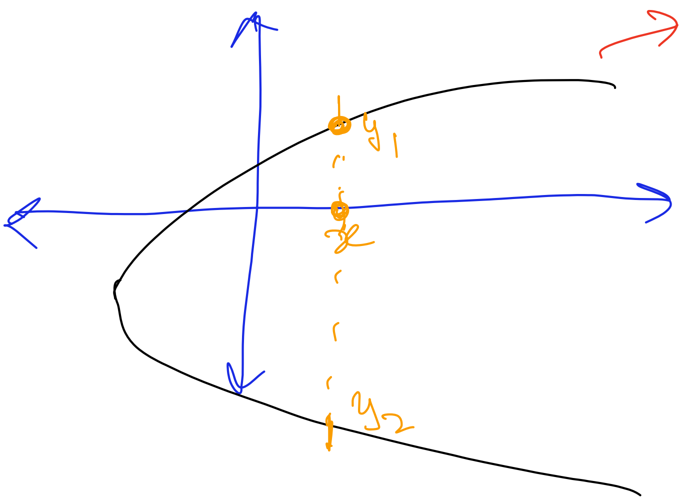


$x=0 \Rightarrow y=1$

$x=1 \Rightarrow y=2$

$x=2 \Rightarrow y=3$

cannot be graph of a function



For an input there are two corresponding outputs

Composition of functions

Given $f(x)$ and $g(x)$

a function — another function

We can define another function, called composition of f and g, as follows:

$$x \longrightarrow \boxed{f} \xrightarrow{f(x)} \boxed{g} \longrightarrow g(f(x))$$

input

the input x goes to the output $g(f(x))$

$$g \circ f\,(x) = g(f(x))$$

new function

Example

$f(x) = x+1$, $g(x) = \sqrt{x}$

Find $g \circ f$.

$$g \circ f\,(0) = g(f(0)) = g(1) = \sqrt{1} = 1$$

$$f(0) = 0+1 = 1$$

$$g\circ f(3) = g(f(3)) = g(3+1) = g(4) = \sqrt{4} = 2$$

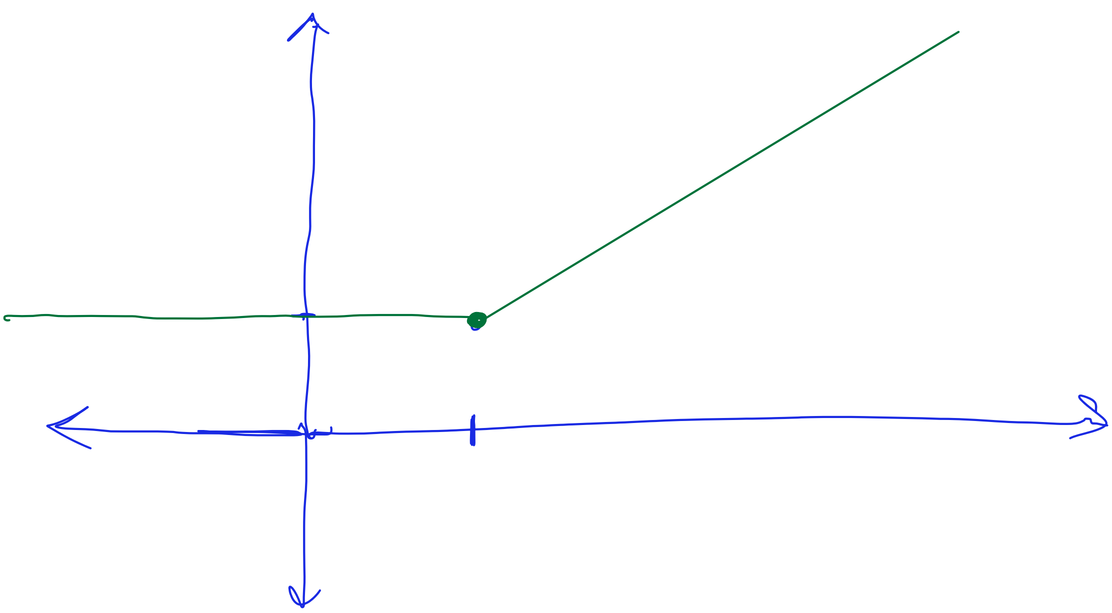
$$g\circ f(x) = g(f(x)) = g(x+1) = \sqrt{x+1}$$

Find $f\circ g$. $f(x) = x+1$, $g(x) = \sqrt{x}$.

$$f\circ g(x) = f(g(x)) = f(\sqrt{x}) = \sqrt{x}+1$$

Compound Functions

$$f(x) = \begin{cases} 1 & x \le 1 \\ x & x > 1 \end{cases}$$



Example

$|\cdot|$ denotes absolute value

$$f(x) = |x|$$

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$