Consider the expression

$$\lim_{x\to 4} \frac{x^2}{x+4} .$$

x:	4.1	4.01	4.001	4.0001	3.9	3.99	3.999	3.9999
$\frac{x^2}{x+4}$:	2.1	2.01	2.001	2.0001	1.9	1.99	1.999	1.9999

$\lim_{x\to 4^+} f(x) = 2$, $\lim_{x\to 4^-} f(x) = 2$

We see that the values of $f(x) = \frac{x^2}{x+4}$ are getting closer and closer to 2 as x approaches 4. We write this as

$$\lim_{x\to 4} \frac{x^2}{x+4} = 2 .$$

Notice that $f(4) = 2$.

$f(4) = \frac{4^2}{4+4} = \frac{16}{8} = 2$

In this case, $\lim_{x\to 4} f(x) = f(4)$

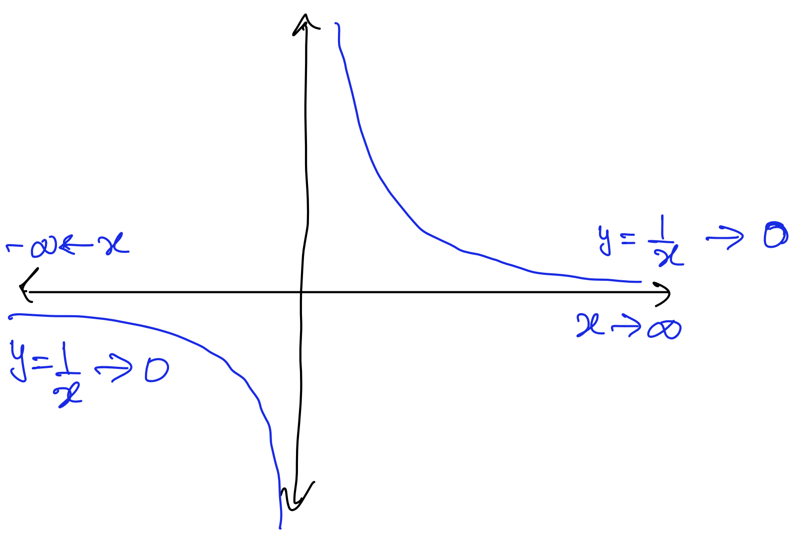
Now consider the limit

$$\lim_{x\to\infty} \frac{1}{x} .$$

As x increases to infinity, its reciprocal $\frac{1}{x}$ decreases to 0. Hence we write

$$\lim_{x\to\infty} \frac{1}{x} = 0 .$$

$\lim_{x\to -\infty} \frac{1}{x} = 0$



Definition of limit
Suppose that $f(x)$ becomes arbitrarily close to the number L (written as $f(x) \to L$) as x approaches a ($x \to a$). Then we say that the limit of $f(x)$ as x approaches a is L and we write

$$\lim_{x \to a} f(x) = L\,.$$

The number a may be replaced by ∞ or $-\infty$.

D.S. (Direct Substitution)

* whenever we have Polynomials or rational functions ($\hookrightarrow \frac{\text{Polyn.}}{\text{Polyn.}}$) just directly substitute $x=a$

Example
Evaluate the following limits:

1. $\lim\limits_{x \to -1} (x^2 - 3)\,.$
2. $\lim\limits_{x \to -2} \dfrac{4 - x^2}{x + 2}\,.$
3. $\lim\limits_{x \to 3} \dfrac{9 - x^2}{x - 3}\,.$

① $\lim\limits_{x \to -1} (x^2 - 3)$

(D.S.) $= (-1)^2 - 3 = 1 - 3 = -2$ (a finite number)

$\Rightarrow \lim\limits_{x \to -1} (x^2 - 3) = -2$

② $\lim\limits_{x \to -2} \dfrac{4 - x^2}{x+2} \overset{\text{D.S.}}{=} \dfrac{4 - (-2)^2}{-2+2} = \dfrac{4-4}{-2+2} = \dfrac{0}{0}$ ← Indeterminate

$= \lim\limits_{x \to -2} \dfrac{2^2 - x^2}{x+2} = \lim\limits_{x \to -2} \dfrac{(2-x)\cancel{(2+x)}}{\cancel{(x+2)}} = \lim\limits_{x \to -2} (2-x)$

Note that $x+2$ cannot be 0 since x cannot be -2. Therefore, $x+2$ can be cancelled.

$\lim\limits_{x \to -2} (2-x) \overset{\text{D.S.}}{=} 2-(-2)$
$= 4$

$\Rightarrow \lim\limits_{x \to -2} \dfrac{4-x^2}{x+2} = 4$

③ $$\lim_{x\to 3} \frac{9-x^2}{x-3} \overset{D.S.}{=} \frac{9-3^2}{3-3} = \frac{9-9}{3-3} = \frac{0}{0}$$

$$\lim_{x\to 3} \frac{3^2-x^2}{x-3} = \lim_{x\to 3} \frac{(3-x)(3+x)}{x-3}$$

$$= \lim_{x\to 3} \frac{-\cancel{(x-3)}(3+x)}{\cancel{(x-3)}} = \lim_{x\to 3} -(3+x)$$

$$\overset{D.S.}{=} -(3+3) = -6$$

One-sided limits

Right hand limit: When x approaches a from the right, that is, through values larger than a, the limit obtained is called right-hand limit and is written as

$$\lim_{x\to a^+} f(x) = L\,.$$

Left hand limit: When x approaches a from the left, that is, through values smaller than a, the limit obtained is called left-hand limit and is written as

$$\lim_{x\to a^-} f(x) = L\,.$$

Example

Evaluate the following limits:

1. $\lim_{x\to 2^+} \sqrt{x-2}\,.$ $\longrightarrow$ well-defined only if $x \geq 2$

   Only $\lim_{x\to 2^+}$ makes sense (RHL)

2. $\lim_{x\to 3^-} \dfrac{3-x}{\sqrt{3-x}}\,.$ $\longrightarrow$ well-defined only if $3-x>0$ $\Rightarrow x<3$

   only $\lim_{x\to 3^-}$ makes sense.

H.W. $\lim_{x\to -3^+} \dfrac{3-x}{\sqrt{3+x}}$

(*) D.S. also works in case of radicals

(1) $\lim_{x\to 2^+} \sqrt{x-2}$

$\underline{\underline{D.S.}}$ $\sqrt{2-2} = 0$

$\Rightarrow \lim_{x\to 2^+} \sqrt{x-2} = 0$

(2) $\lim_{x\to 3^-} \dfrac{3-x}{\sqrt{3-x}}$ $\underline{\underline{D.S}}$ $\dfrac{3-3}{\sqrt{3-3}} = \dfrac{0}{0}$

* $(\sqrt{3-x})^2 = 3-x$

$= \lim_{x\to 3^-} \dfrac{(\sqrt{3-x})^{\cancel{2}}}{\cancel{\sqrt{3-x}}}$

$\dfrac{a^2}{a} = \dfrac{a\times\cancel{a}}{\cancel{a}} = a$

$= \lim_{x\to 3^-} \sqrt{3-x} = \sqrt{3-3} = 0$

Properties of limits

Let $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then we have

1. $\lim_{x\to a}[f(x) \pm g(x)] = L \pm M$.
2. $\lim_{x\to a} f(x)g(x) = LM$.
3. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$ provided that $M \neq 0$.
4. $\lim_{x\to a} kf(x) = kL$.

can be +, − or 0

$$\frac{\text{any finite number}}{\infty} = 0$$

Example

Evaluate the limit

$$\lim_{x\to\infty} \frac{3x^2 + x + 1}{2x^2 - x + 2} .$$

D.S. $\frac{\infty}{\infty}$

$$= \lim_{x\to\infty} \frac{\frac{3x^2+x+1}{x^2}}{\frac{2x^2-x+2}{x^2}} = \lim_{x\to\infty} \frac{\frac{3x^2}{x^2} + \frac{x}{x^2} + \frac{1}{x^2}}{\frac{2x^2}{x^2} - \frac{x}{x^2} + \frac{2}{x^2}}$$

$$\boxed{\lim_{x\to\infty} \frac{1}{x} = 0}$$

$$= \lim_{x\to\infty} \frac{3 + \frac{1}{x} + \frac{1}{x^2}}{2 - \frac{1}{x} + \frac{2}{x^2}}$$

$$= \frac{\lim_{x\to\infty}\left(3 + \frac{1}{x} + \frac{1}{x^2}\right)}{\lim_{x\to\infty}\left(2 - \frac{1}{x} + \frac{2}{x^2}\right)}$$

$$= \frac{\left(\lim_{x\to\infty} 3\right) + \left(\lim_{x\to\infty} \frac{1}{x}\right) + \left(\lim_{x\to\infty} \frac{1}{x^2}\right)}{\left(\lim_{x\to\infty} 2\right) - \left(\lim_{x\to\infty} \frac{1}{x}\right) + \left(\lim_{x\to\infty} \frac{2}{x^2}\right)} = \frac{3+0+0}{2-0+0}$$

$$= \frac{3}{2}$$

Example

$$\lim_{x\to\infty} \frac{3-x}{x^2+1} \begin{matrix}\to 1\\ \to 2\end{matrix} \Big\} \text{ divide by } x^2$$

$$= \lim_{x\to\infty} \frac{\frac{3}{x^2} - \frac{x}{x^2}}{\frac{x^2}{x^2} + \frac{1}{x^2}} = \lim_{x\to\infty} \frac{\frac{3}{x^2} - \frac{1}{x}}{1 + \frac{1}{x^2}}$$

$$= \frac{\lim_{x\to\infty}\left(\frac{3}{x^2} - \frac{1}{x}\right)}{\lim_{x\to\infty}\left(1 + \frac{1}{x^2}\right)} = \frac{0-0}{1+0} = \frac{0}{1} = 0$$

(*) $\lim_{x\to\infty} \frac{p(x)}{q(x)} = 0$ if $\deg p < \deg q$

Let $\deg p = n < \deg q = m$

$n < m \Rightarrow$ divide by x^m

$$p(x) = a_0 + a_1 x + \ldots + a_n x^n$$

$$\frac{p(x)}{x^m} = \frac{a_0}{x^m} + \frac{a_1 x}{x^m} + \ldots + \frac{a_n x^n}{x^m}$$

$n < m$

(*) $\lim_{x\to\infty} \frac{p(x)}{q(x)} = \infty$ if $\deg p > \deg q$

6. $\lim_{x\to\infty}\left(\sqrt{x^2+4}-x\right)$. Hint: multiply by $\dfrac{\sqrt{x^2+4}+x}{\sqrt{x^2+4}+x}$.

$$\lim_{x\to\infty}\left(\sqrt{x^2+4}-x\right)$$

$$=(\infty-\infty) \quad [\text{Indeterminate}]$$

$$\Rightarrow \lim_{x\to\infty}\left(\sqrt{x^2+4}-x\right)\frac{\left(\sqrt{x^2+4}+x\right)}{\left(\sqrt{x^2+4}+x\right)}$$

$$=\lim_{x\to\infty}\frac{\left(\sqrt{x^2+4}-x\right)\left(\sqrt{x^2+4}+x\right)}{\sqrt{x^2+4}+x}$$

$$=\lim_{x\to\infty}\frac{\left(\sqrt{x^2+4}\right)^2-x^2}{\sqrt{x^2+4}+x}$$

$$=\lim_{x\to\infty}\frac{x^2+4-x^2}{\sqrt{x^2+4}+x}=\lim_{x\to\infty}\frac{4}{\sqrt{x^2+4}+x}$$

$$=\frac{4}{\infty+\infty}=\frac{4}{\infty}=0$$

Definition of continuity
A function f is continuous at $x = a$ if f is defined at a and

$$\lim_{x\to a} f(x) = f(a)\,.$$

If a function f is continuous at all points in an interval, it is said to be continuous in the interval.

Example Find whether the following functions are continuous at $x = 1$.

1. $f(x) = x^2 + x$.
2. $g(x) = \begin{cases} 2 & x \geq 1 \\ x^3 & x < 1 \end{cases}$.

- Find $\lim_{x\to 1} f(x) = L$
- Check whether $L = f(1)$ or not.

① $\lim_{x\to 1} f(x) = 1^2 + 1 = 2 = f(1)$

$\Rightarrow f$ is continuous at $x=1$

② $\text{LHL}_{x=1} = \lim_{x\to 1} x^3 = 1^3 = 1$

$\text{RHL}_{x=1} = \lim_{x\to 1} 2 = 2$

$\text{LHL}_{x=1} \neq \text{RHL}_{x=1} = f(1)$

$\Rightarrow g$ is not continuous at $x=1$. (discontinuous)

Examples of Continuous fns.

- Polynomials are cont. over the entire real line
  $y = x^n$, $n > 0$
- rational functions in their domains are cont.
- radicals are also cont. in their domain.
- Sums, difference and Products of cont. fns. are cont.
- ratio of two cont. fns. is cont. as long as the denominator does not become 0.