The derivative as slope of tangent (section 2.3)

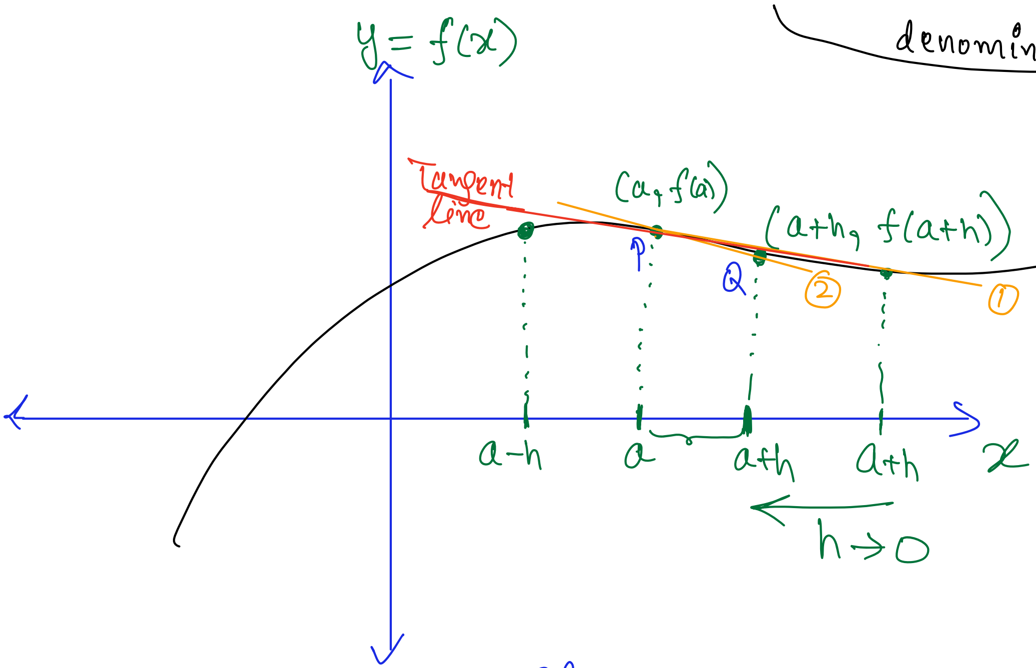
$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

or

numerator $= \Delta y$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

denominator $\to h = \Delta x$



Slope of secant line = Slope of line joining $(a, f(a))$, $(a+h, f(a+h))$

$$m_{PQ} = \frac{f(a+h) - f(a)}{a+h-a}$$

$$= \frac{f(a+h) - f(a)}{h}$$

$$\lim_{h \to 0} m_{PQ} = m_{\text{Tangent}} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$\uparrow$
a

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

The Four Step Process (section 2.4)

1. Replace x by $x+h$ in the equation $y = f(x)$ to obtain $y + \Delta y = f(x+h)$.
2. Subtract $y = f(x)$ from both sides to obtain $\Delta y = f(x+h) - f(x)$.
3. Divide both sides by $\Delta x = h$ to obtain $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x+h)-f(x)}{h}$.
4. Take the limit $\Delta x = h \to 0$ on both sides to obtain

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\,.$$

Note: The derivative $f'(x)$ gives us the instantaneous rate of change of one variable with respect to another.

Example 1.
Find the derivative of the following functions:

1. $y = \dfrac{1}{x-1}$.
2. $y = \sqrt{x+2}$.

Input Output

$x \longmapsto f'(x)$ or $\dfrac{dy}{dx}$ or $\dfrac{d}{dx}(f(x))$

varying over reals

① $f(x) = \dfrac{1}{x-1}$

Step 1: replace x by $x+h$

$$f(x+h) = \frac{1}{x+h-1}$$

Step 2: $\Delta y = f(x+h) - f(x) = \dfrac{1}{x+h-1} - \dfrac{1}{x-1}$

$$= \frac{x-1-(x+h-1)}{(x+h-1)(x-1)}$$

$$= \frac{\cancel{x}-\cancel{1}-\cancel{x}-h+\cancel{1}}{(x+h-1)(x-1)}$$

$$= \frac{-h}{(x+h-1)(x-1)}$$

Step 3 $\quad \dfrac{f(x+h)-f(x)}{h} = \dfrac{1}{h} \cdot \dfrac{-h}{(x+h-1)(x-1)}$

$$= \frac{-1}{(x+h-1)(x-1)}$$

Step 4 $\quad \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{-1}{(x+h-1)(x-1)}$

$$= \frac{-1}{(x-1)(x-1)} = \frac{-1}{(x-1)^2}$$

$$f'(x) = \frac{-1}{(x-1)^2}.$$

(2) $\quad f(x) = \sqrt{x+2}$

Step 1 $\quad f(x+h) = \sqrt{x+h+2}$

Step 2 $\quad f(x+h) - f(x) = \sqrt{x+h+2} - \sqrt{x+2}$

Step 3 $\quad \dfrac{f(x+h)-f(x)}{h} = \dfrac{\sqrt{x+h+2} - \sqrt{x+2}}{h}$

Step 4 $\quad f'(x) = \lim_{h\to 0} \dfrac{\sqrt{x+h+2} - \sqrt{x+2}}{h}$

Direct subs $\xrightarrow{h=0}$ $\dfrac{\sqrt{x+2} - \sqrt{x+2}}{0} = \dfrac{0}{0}$

$$f'(x) = \lim_{h \to 0} \frac{\left(\sqrt{x+h+2} - \sqrt{x+2}\right)}{h} \cdot \frac{\left(\sqrt{x+h+2} + \sqrt{x+2}\right)}{\left(\sqrt{x+h+2} + \sqrt{x+2}\right)}$$

$$= \lim_{h \to 0} \frac{\left(\sqrt{x+h+2}\right)^2 - \left(\sqrt{x+2}\right)^2}{h\left(\sqrt{x+h+2} + \sqrt{x+2}\right)}$$

$$= \lim_{h \to 0} \frac{x+h+2 - (x+2)}{h\left(\sqrt{x+h+2} + \sqrt{x+2}\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{x}+h+\cancel{2} - \cancel{x} - \cancel{2}}{h\left(\sqrt{x+h+2} + \sqrt{x+2}\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h+2} + \sqrt{x+2}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h+2} + \sqrt{x+2}}$$

D.S.

$$= \frac{1}{\sqrt{x+2} + \sqrt{x+2}} = \frac{1}{2\sqrt{x+2}} = f'(x)$$

$$\boxed{\circledast \; D_{f'} = \{x+2 > 0\} = \{x : x > -2\}} \quad (-2, \infty)$$

Derivatives of polynomials (section 2.5)

1. The constant rule: $\frac{dc}{dx} = 0$ where c is a constant.
2. The derivative of x: $\frac{dx}{dx} = 1$.
3. The derivative of x^n, $n > 0$: $\frac{dx^n}{dx} = nx^{n-1}$.
4. The sum rule: $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$.
5. The constant multiple rule: $\frac{d}{dx}(cu) = c\frac{du}{dx}$.

$$\frac{d}{dx}(x^3+x^2) = \frac{d}{dx}(x^3) + \frac{d}{dx}(x^2) = 3x^2 + 2x$$

$$\frac{d}{dx}(2x^3 - x^2) = 2\frac{d}{dx}(x^3) - \frac{d}{dx}(x^2) = 2(3x^2) - 2x = 6x^2 - 2x$$

① $f(x) = c$

Step 1 $f(x+h) = c$

Step 2 $f(x+h) - f(x) = c - c = 0$

Step 3 $\frac{f(x+h) - f(x)}{h} = \frac{0}{h} = 0$

Step 4 $f'(x) = \lim_{h \to 0} 0 = 0$

② $f(x) = x$

Step 1 $f(x+h) = x+h$

Step 2 $f(x+h) - f(x) = x+h - x = h$

Step 3 $\frac{f(x+h) - f(x)}{h} = \frac{h}{h} = 1$

Step 4 $f'(x) = \lim_{h \to 0} 1 = 1$

③ $f(x) = x^n$ (n here is a Positive integer)

$\Rightarrow f'(x) = n x^{n-1}$

$\boxed{n=2}$ $f(x) = x^2$

Step 1 $f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$

Step 2 $f(x+h) - f(x) = \cancel{x^2} + 2xh + h^2 - \cancel{x^2}$

$= 2xh + h^2$

Step 3 $\dfrac{f(x+h) - f(x)}{h} = \dfrac{2xh + h^2}{h} = \dfrac{\cancel{h}(2x+h)}{\cancel{h}}$

$= 2x + h$

Step 4 $f'(x) = \lim_{h \to 0} (2x+h) = 2x$

$\boxed{n=3}$ $f(x) = x^3$

Step 1 $f(x+h) = (x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$

For general n : $f(x+h) = (x+h)^n$

$= x^n + n x^{n-1} h + \binom{n}{2} x^{n-2} h^2 + \dots + h^n$

$f'(x)$

Q $\dfrac{d}{dx}\left(x^{1000}\right) = 1000\, x^{999}$

• $\dfrac{d}{dx}\left(x^{10}\right) = 10x^9$

Example 2. Differentiate the following:

1. $y = 3x^4$.
2. $y = 4x^5 + 5x^3 - x^2 + 1$.

1 $\frac{dy}{dx} = \frac{d}{dx}(3x^4) = 3\frac{d}{dx}(x^4) = 3(4x^{4-1})$

$= 3(4x^3)$

$= 12x^3$

2 $\frac{dy}{dx} = 4(x^5)' + 5(x^3)' - (x^2)' + (1)'$

$= 4(5x^4) + 5(3x^2) - 2x + 0$

$= 20x^4 + 15x^2 - 2x$

$\frac{dy}{dx} = y'$