The product rule:

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Example 1. Find the derivative of $y = (x^3 + 2x^2 - 3x)(x^3 - 4x)$.

$$\Rightarrow y' = (x^3+2x^2-3x)(x^3-4x)' + (x^3+2x^2-3x)'(x^3-4x)$$

$$= (x^3+2x^2-3x)(3x^2-4) + (3x^2+4x-3)(x^3-4x)$$

$$(x-1)(x+2) + (x-1)$$

$$(x-1)(x+2+1)$$

$$(x-1)(x+3)$$

The quotient rule:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Example 2. Find the derivative of $y = \frac{x^3 - 3x}{x + 1}$.

$$\frac{dy}{dx} = \frac{(x+1)(x^3-3x)' - (x^3-3x)(x+1)'}{(x+1)^2}$$

$$= \frac{(x+1)(3x^2-3) - (x^3-3x)}{(x+1)^2}$$

only if asked to simplify

$$= \frac{3x^3 - \cancel{3x} + 3x^2 - 3 - x^3 + \cancel{3x}}{(x+1)^2} = \frac{2x^3+3x^2-3}{(x+1)^2}$$

Chain rule:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

The generalized power rule:

$$\frac{du^n}{dx} = nu^{n-1}\frac{du}{dx}.$$

Example 3. Differentiate $f(x) = \sqrt{x^2+1}$.

$$f'(x) = \frac{d}{dx}\left(\sqrt{x^2+1}\right) = \frac{d}{dx}\left[\left(\underbrace{x^2+1}_{u}\right)^{\overset{n}{1/2}}\right]$$

$$= \frac{1}{2}\left(x^2+1\right)^{\frac{1}{2}-1}\frac{d}{dx}\left(x^2+1\right)$$

$$= \frac{1}{2}\left(x^2+1\right)^{-1/2}(2x) = \frac{\cancel{2}x}{\cancel{2}}\frac{1}{\left(x^2+1\right)^{1/2}}$$

$$= \boxed{\frac{x}{\sqrt{x^2+1}}}$$

Example 4. Let $y = x\sqrt{x^2-2x}$. Find y'.

$$\frac{dy}{dx} = x\left(\sqrt{x^2-2x}\right)' + (x)'\left(\sqrt{x^2-2x}\right)$$

$$= x\left(\sqrt{\underbrace{x^2-2x}_{u}}\right)' + \sqrt{x^2-2x} \qquad n = 1/2$$

$$\left(\sqrt{x^2-2x}\right)' = \left[\left(\underbrace{x^2-2x}_{u}\right)^{\overset{n}{1/2}}\right]' = \frac{1}{2}\left(x^2-2x\right)^{\frac{1}{2}-1}\frac{d}{dx}\left(x^2-2x\right)$$

$$= \frac{1}{2\sqrt{x^2-2x}}(2x-2) = \frac{\cancel{2}(x-1)}{\cancel{2}\sqrt{x^2-2x}} = \frac{x-1}{\sqrt{x^2-2x}}$$

$$\frac{dy}{dx} = x\,\frac{(x-1)}{\sqrt{x^2-2x}} + \sqrt{x^2-2x}$$

$$= \frac{x(x-1) + x^2-2x}{\sqrt{x^2-2x}}$$

$$= \frac{x^2-x+x^2-2x}{\sqrt{x^2-2x}}$$

$$\frac{dy}{dx} = \frac{2x^2-3x}{\sqrt{x^2-2x}} = \frac{x(2x-3)}{\sqrt{x^2-2x}}$$

Example 5. Find the derivative of $y = \frac{x}{\sqrt{x^2+1}}$.

$$\frac{dy}{dx} = \frac{\sqrt{x^2+1}\,(x)' - x\,(\sqrt{x^2+1})'}{(\sqrt{x^2+1})^2} \quad \text{(Quotient rule)}$$

$$= \frac{\sqrt{x^2+1} - x\,(\sqrt{x^2+1})'}{x^2+1}$$

$$(\sqrt{x^2+1})' = \frac{x}{\sqrt{x^2+1}} \quad \text{(from Example 3)}$$

$$\frac{dy}{dx} = \frac{\sqrt{x^2+1} - x\left(\frac{x}{\sqrt{x^2+1}}\right)}{x^2+1}$$

$$= \frac{\sqrt{x^2+1} - \frac{x^2}{\sqrt{x^2+1}}}{x^2+1}$$

$$= \frac{\frac{\cancel{x^2}+1-\cancel{x^2}}{\sqrt{x^2+1}}}{(x^2+1)} = \frac{1}{\sqrt{x^2+1}\,(x^2+1)}$$

$$= \frac{1}{(x^2+1)^{3/2}}$$

Note. To find the derivative of functions of the form

$$y = \frac{k}{g(x)},$$

write $y = k[g(x)]^{-1}$ and use generalized power rule (instead of quotient rule).

Example 6. Differentiate $y = \dfrac{4}{\sqrt[3]{x^3 + x}}$.

$$y = \frac{4}{(x^3+x)^{1/3}} = 4\left[(x^3+x)^{1/3}\right]^{-1}$$

$$= 4(x^3+x)^{-1/3}$$

$$\frac{dy}{dx} = 4\frac{d}{dx}\left[(\underbrace{x^3+x})^{-1/3}\right] = 4\left(\frac{-1}{3}\right)(x^3+x)^{\frac{-1}{3}-1}\cdot(x^3+x)'$$

$$= \frac{-4}{3}(x^3+x)^{-4/3}(3x^2+1)$$

$$= \frac{-4}{3}\frac{3x^2+1}{(x^3+x)^{4/3}}$$