Example 1. Find the slope of tangent to the circle $x^2 + y^2 = 25$ at the point $(4, -3)$.

$x^2+y^2=25$ → Not in the form $y=f(x)$

$y^2 = 25-x^2$

$y = \pm\sqrt{25-x^2}$

Since $-3<0$, we choose $y=-\sqrt{25-x^2}$

y coordinate of $(4,-3)$

can get $\frac{dy}{dx}$ (involves more calculations)

→ Easier to evaluate $\frac{dy}{dx}$ using implicit diff.

$x^2+y^2=25$

$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$

$2x + \frac{d}{dx}(y^2) = 0$

$2x + 2y\left(\frac{dy}{dx}\right) = 0$

$2y\left(\frac{dy}{dx}\right) = -2x$

$\Rightarrow \frac{dy}{dx} = \frac{-2x}{2y}$

At $(4,-3)$, $y=-3$, $x=4$ $\Rightarrow \frac{dy}{dx} = \frac{-\cancel{2}(4)}{\cancel{2}(-3)} = \frac{4}{3}$

slope of tangent at $(4,-3)$

Example 2. Find $\frac{dy}{dx}$ for each of the following equations:

1. $3y^2 = x$.

$$\Rightarrow \frac{d}{dx}(3y^2) = \frac{d}{dx}(x)$$

$$\Rightarrow 3\frac{d}{dx}(y^2) = 1$$

$$\Rightarrow 6y\left(\frac{dy}{dx}\right) = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{6y}$$

2. $x^2y^2 = 1$.

$$\frac{d}{dx}(x^2y^2) = \frac{d}{dx}(1)$$

$$\Rightarrow \left[\frac{d}{dx}(x^2)\right]y^2 + x^2\left[\frac{d}{dx}(y^2)\right] = 0$$

$$\Rightarrow 2xy^2 + x^2\left[2y\left(\frac{dy}{dx}\right)\right] = 0$$

$$\Rightarrow 2xy^2 + (2x^2y)\left(\frac{dy}{dx}\right) = 0 \Rightarrow (2x^2y)\left(\frac{dy}{dx}\right) = -2xy^2$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\cancel{2x}y^{\cancel{2}}}{\cancel{2}x^{\cancel{2}}\cancel{y}}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y}{x}$$

3. $x^2y = 4$.

$$\frac{d}{dx}(x^2y) = \frac{d}{dx}(4)$$

$$\Rightarrow \left[\frac{d}{dx}(x^2)\right]y + x^2\left[\frac{dy}{dx}\right] = 0$$

$$\Rightarrow 2xy + x^2\left(\frac{dy}{dx}\right) = 0 \Rightarrow x^2\left(\frac{dy}{dx}\right) = -2xy$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2\cancel{x}y}{x^{\cancel{2}}} = -\frac{2y}{x}$$

Example 3. Find dy/dx implicitly if $3x^2 + 4x^3y^4 + 2y = 4$.

$$\Rightarrow \frac{d}{dx}(3x^2) + \frac{d}{dx}(4x^3y^4) + \frac{d}{dx}(2y) = \frac{d}{dx}(4)$$

$$\Rightarrow 6x + \left[\frac{d}{dx}(4x^3)\right]y^4 + 4x^3\left[\frac{d}{dx}(y^4)\right] + 2\left(\frac{dy}{dx}\right) = 0$$

$$\Rightarrow 6x + 12x^2y^4 + 4x^3\left[4y^3\left(\frac{dy}{dx}\right)\right] + 2\left(\frac{dy}{dx}\right) = 0$$

$$\Rightarrow 6x + 12x^2y^4 + 16x^3y^3\left(\frac{dy}{dx}\right) + 2\left(\frac{dy}{dx}\right) = 0$$

$$\Rightarrow (6x + 12x^2y^4) + (16x^3y^3 + 2)\left(\frac{dy}{dx}\right) = 0$$

$$\Rightarrow (16x^3y^3 + 2)\left(\frac{dy}{dx}\right) = -(6x + 12x^2y^4) \Rightarrow \frac{dy}{dx} = \frac{-6x - 12x^2y^4}{16x^3y^3 + 2}$$

$$= \frac{-3x - 6x^2y^4}{8x^3y^3 + 1}$$

Example 4. Find the slope of tangent to the graph of the equation

$$x^2 - 3xy + y^2 + 4x - 2y = 1\,,$$

at the point $(1, 4)$. $\longrightarrow \left.\frac{dy}{dx}\right|_{\substack{x=1\\ y=4}}$

$$\Rightarrow \frac{d}{dx}(x^2) - 3\left[\frac{d}{dx}(xy)\right] + \frac{d}{dx}(y^2) + 4 - 2\left(\frac{dy}{dx}\right) = 0$$

$$\Rightarrow 2x - 3\left[\left(\frac{d}{dx}(x)\right)y + x\left(\frac{dy}{dx}\right)\right] + 2y\left(\frac{dy}{dx}\right) + 4 - 2\left(\frac{dy}{dx}\right) = 0$$

$$\Rightarrow 2x - 3\left[y + x\left(\frac{dy}{dx}\right)\right] + 2y\left(\frac{dy}{dx}\right) + 4 - 2\left(\frac{dy}{dx}\right) = 0$$

$$\Rightarrow 2x - 3y - 3x\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) + 4 - 2\left(\frac{dy}{dx}\right) = 0$$

$$\Rightarrow (2x - 3y + 4) + (-3x + 2y - 2)\left(\frac{dy}{dx}\right) = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(2x - 3y + 4)}{-3x + 2y - 2} \Rightarrow \text{slope} = -\frac{(2 - 12 + 4)}{-3 + 8 - 2} = \frac{6}{3}$$

$= 2$

Example 5. Find the equations of the tangent and normal lines to the parabola $y^2 = 3x$ at the point $(3, -3)$.

Find slope of tangent m_T :-

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(3x) \Rightarrow 2y\left(\frac{dy}{dx}\right) = 3$$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{2y}$$

At $(3,-3)$, $\left.\frac{dy}{dx}\right|_{\substack{x=3\\y=-3}} = \frac{3}{2(-3)} = \frac{-1}{2}$

$$\Rightarrow m_T = \frac{-1}{2}$$

slope of tangent.

Find eqn. of tangent :-

$$\frac{y-y_1}{x-x_1} = m_T \Rightarrow \frac{y-(-3)}{x-3} = \frac{-1}{2}$$

$$\Rightarrow \frac{y+3}{x-3} = \frac{-1}{2} \Rightarrow 2(y+3) = -(x-3)$$

$$\Rightarrow 2y+6 = -x+3$$

$$\Rightarrow x+2y+3=0$$

eqn of tangent.

Equation of normal :-

Normal is perpendicular to the tangent.

$\Rightarrow$ slope of normal $m_N = \frac{-1}{m_T} = \frac{-1}{-\frac{1}{2}} = 2$

$$\Rightarrow \quad \frac{y+3}{x-3} = 2 \quad \Rightarrow \quad y+3 = 2x-6$$

$$\Rightarrow \quad 2x - y - 9 = 0$$