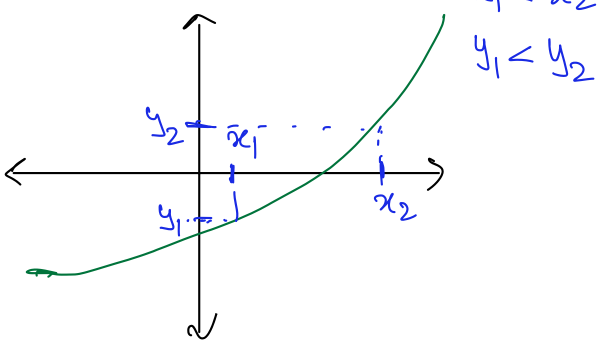
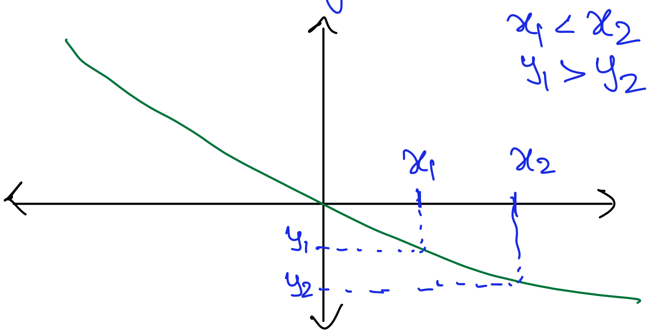
Increasing and Decreasing Functions:

1. A function f is increasing on an interval if, for any two numbers x_1 and x_2 in the interval,
$$\boxed{x_1 < x_2 \text{ implies that } f(x_1) < f(x_2)\,.} \Rightarrow f'(x) > 0$$
2. A function f is decreasing on an interval if, for any two numbers x_1 and x_2 in the interval,
$$\boxed{x_1 < x_2 \text{ implies that } f(x_1) > f(x_2)\,.} \Rightarrow f'(x) < 0$$

Increasing Function



Decreasing Function



Example 1. Let $y = 1 - x^2$. Determine the intervals in which the function is increasing and decreasing.

$$\Rightarrow \frac{dy}{dx} = -2x$$

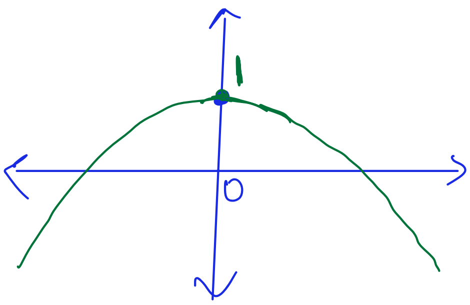
- when is $\frac{dy}{dx} > 0$:—

$$-2x > 0 \Rightarrow \frac{-2x}{-2} < \frac{0}{-2}$$

$$\Rightarrow \underbrace{x < 0}_{\text{Interval where } f \text{ is increasing}} \equiv (-\infty, 0)$$

- when is $\frac{dy}{dx} < 0$:—

$$-2x < 0 \Rightarrow \underbrace{x > 0}_{\text{Interval where } f \text{ is decreasing}} \equiv (0, \infty)$$



Relative maximum and minimum:

1. A point is called a relative (or local) maximum if it has a larger y-value than any point near it.
2. A point is called a relative (or local) minimum if it has a smaller y-value than any point near it.
3. Either of maximum and minimum points/values are called extreme points/values.

extremal Points.

Example 2. Determine the intervals on which the function $y = x^3 - 3x + 2$ is increasing and decreasing. From this information determine the maximum and minimum points.

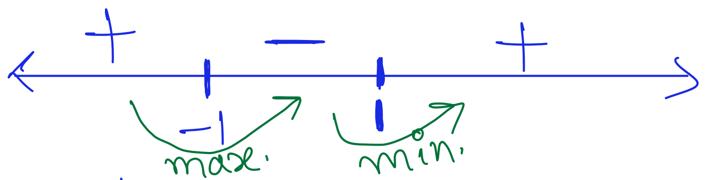
$$y' = 3x^2 - 3 = 3(x^2 - 1)$$

- when is $y' > 0$:-

$$3(x^2-1) > 0 \Rightarrow x^2 - 1 > 0$$

Factorize

$$(x-1)(x+1)$$



$y' > 0$ when x lies in $\underbrace{(-\infty,-1) \cup (1,\infty)}_{\text{Increasing}}$

$y' < 0$ when x lies in $\underbrace{(-1,1)}_{\text{decreasing}}$

max. when $f'(x)$ changes from being $+$ to $-$

min. when $f'(x)$ changes from being $-$ to $+$

$\Rightarrow$ $x = -1$ is a point of local maximum

and $x = 1$ is a point of local minimum.

Critical Points: A number c in the domain of f for which $f'(c) = 0$ is called a critical number of f. The points $(c, f(c))$ are called critical points.

First Derivative Test:

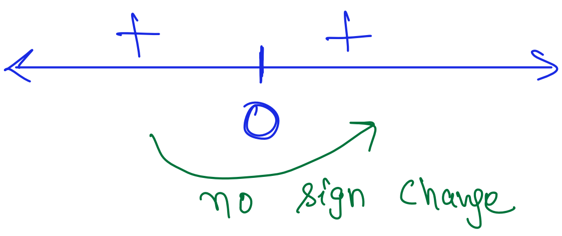
1. Find the critical numbers of f. Suppose c is a critical number.
2. Test the derivative with two values of x, one slightly less and the other slightly more than c.
   (a) If, as x increases, the sign of the derivative changes from $+$ to $-$, then $f(c)$ is a maximum value and $(c, f(c))$ is a maximum point.
   (b) If the sign changes from $-$ to $+$, then $f(c)$ is a minimum value.
   (c) If the sign does not change, then $(c, f(c))$ is neither a minimum nor a maximum.

Example 3. Test the function $f(x) = x^3$ for extreme values and sketch the graph.

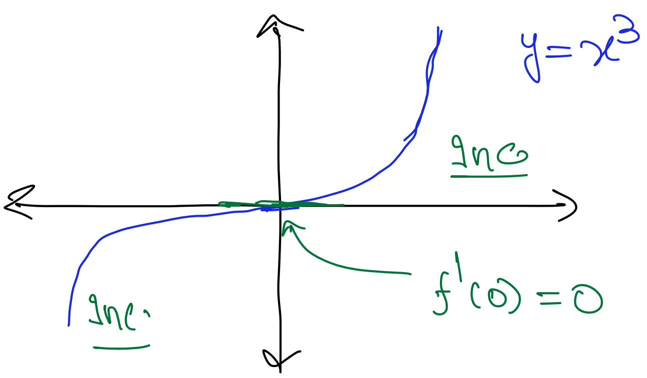
$\Rightarrow f'(x) = 3x^2$

Find critical numbers :-

$f'(x) = 0 \Rightarrow 3x^2 = 0 \Rightarrow x^2 = 0 \Rightarrow x = 0$



$\Rightarrow x = 0$ is neither a maximum pt. nor a minimum pt.



Example 4. Find the extreme values of the function $y = (-2/3)x^3 + x^2 + 4x - 5$ and sketch the graph.

$$y' = \frac{-2}{3}(3x^2) + 2x + 4$$

$$\Rightarrow y' = -2x^2 + 2x + 4$$

$$= -2(\underbrace{x^2 - x - 2}_{\text{factorize}})$$

Find two numbers whose product is -2 and sum is -1.

Middle term splitting

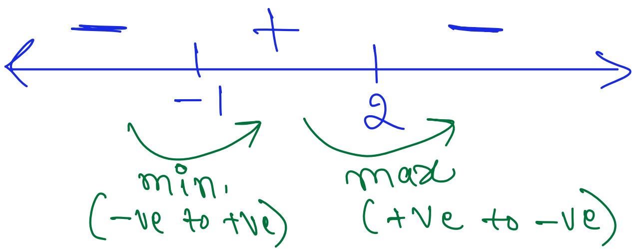
$$x^2 - x - 2 = x^2 - 2x + x - 2$$
$$= x(x-2) + 1(x-2)$$
$$= (x-2)(x+1)$$

$$\left[x^2 - x - 2 = (x-2)(x+1)\right]$$

$$y' = -2(x-2)(x+1) = 0$$

[Find critical numbers
Put $f'(x) = 0$]

$$x - 2 = 0 \Rightarrow x = 2 \quad \text{and} \quad x + 1 = 0 \Rightarrow x = -1$$



$y(-1)$ is min. value

$y(2)$ is max. value

⊛ We look for sign change of $f'(x)$.

In this example, $f'(x) = -2(x^2 - x - 2)$

$\Rightarrow$ $x = -1$ is a min point

and $x = 2$ is a max point

To find min value, plug $x = -1$ in $\boxed{f(x) \text{ or } y}$ Not $f'(x)$

To find max value, plug $x = 2$ in $\boxed{f(x) \text{ or } y}$ Not $f'(x)$

$\Rightarrow$ min value $= f(-1) = -\frac{2}{3}(-1)^3 + (-1)^2 + 4(-1) - 5$

$= \frac{2}{3} + 1 - 4 - 5 = -\frac{22}{3}$

$\Rightarrow$ max value $= f(2) = -\frac{2}{3}(2)^3 + (2)^2 + 4(2) - 5$

$= -\frac{16}{3} + 4 + 8 - 5 = \frac{5}{3}$

Note that

The max./min here are relative.

It may happen sometimes that the min value comes out to be greater than max value.

We determine min/max from how $f'(x)$ is changing its sign about a critical point and NOT by comparing the final values of f on the critical points.