Concavity and Inflection Points: For every x lying in an interval $[a, b]$, the graph of a function f is:

1. concave up if $f''(x) > 0$,
2. concave down if $f''(x) < 0$.

A point on the graph at which the concavity changes is called an inflection point.

Example 1. Find the inflection points and graph the function $y = x^3 - 3x + 2$ of Example 2 from the previous lecture.

→ Find the second derivative.

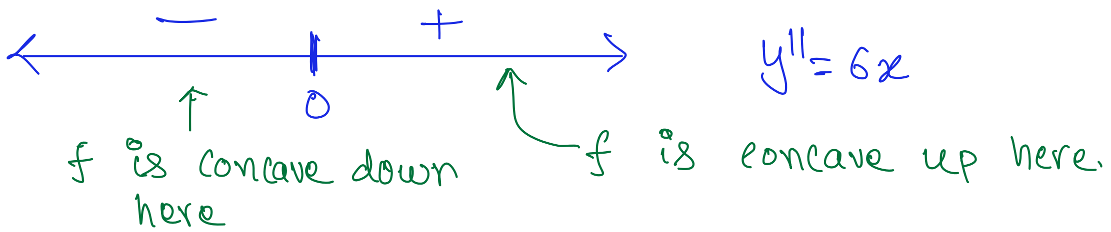
$$y' = 3x^2 - 3$$

$$y'' = 6x$$

→ Put $y'' = 0$

$\Rightarrow 6x = 0 \Rightarrow x = 0$

→ Draw number line and locate $x = 0$.
Then find whether the sign of y'' changes about $x = 0$.



→ Since y'' does change sign about $x = 0$, it is an inflection point.

Extremal points

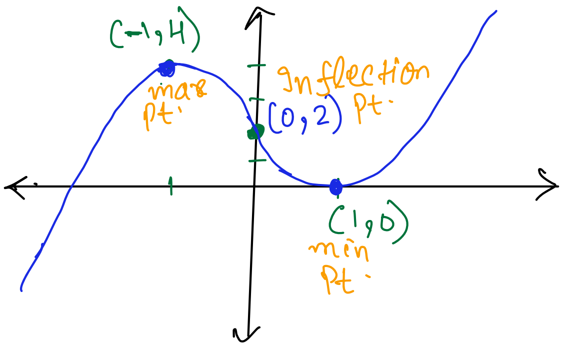
From Example 2 of previous lecture, we have:–

$x = -1$ is max point and $x = 1$ is min point.

max value $= f(-1) = (-1)^3 - 3(-1) + 2 = 4$

min value $= f(1) = 1^3 - 3(1) + 2 = 0$.

Graph



Second Derivative Test:

1. Find the critical numbers of f. Suppose c is a critical number.
2. Find $f''(c)$.
   (a) If $f''(c) > 0$ then $f(c)$ is a minimum value and $(c, f(c))$ is a minimum point.
   (b) If $f''(c) < 0$, then $f(c)$ is a maximum value.
   (c) If $f''(c) = 0$, then the test fails and $(c, f(c))$ may be a relative minimum or a relative maximum or neither.

Example 2. Test the function $f(x) = x^3 - 3x$ for extreme values.

- Find the 2nd derivative of f.

$\Rightarrow f'(x) = 3x^2 - 3$

$\Rightarrow f''(x) = 6x$

- Find critical numbers by solving $f'(x) = 0$

$\Rightarrow 3x^2 - 3 = 0 \Rightarrow 3x^2 = 3 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

$\Rightarrow$ we have two critical numbers : $x = 1$, $x = -1$.

- For each critical number c, find $f''(c)$.

$\Rightarrow f''(1) = 6(1) = 6 \Rightarrow x = 1$ is a minimum pt.

$\Rightarrow f''(-1) = 6(-1) = -6 \Rightarrow x = -1$ is a maximum pt

- Find the max and min values

max value $= f(-1) = (-1)^3 - 3(-1) = 2$

Min Value $= f(1) = (1)^3 - 3(1) = -2$

Procedure for Curve Sketching:

1. Find all critical numbers.
2. Test the critical numbers for relative extremal points.
   - (a) Use the second derivative test.
   - (b) If the second derivative test fails, use the first derivative test.
3. Use the second derivative test to determine intervals where graph of f is concave up and where it is concave down.
4. Determine the points of inflection.
5. Find any easily determined intercepts.
6. Plot the critical points, inflection points and intercepts.
7. Sketch an approximate curve.

find max/min

Inflection Pts

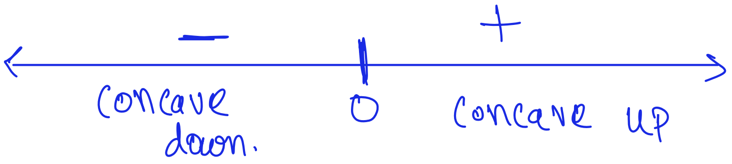
$y = 0$ (x int.) can be more than one

$x = 0$ (y-int.) atmost one

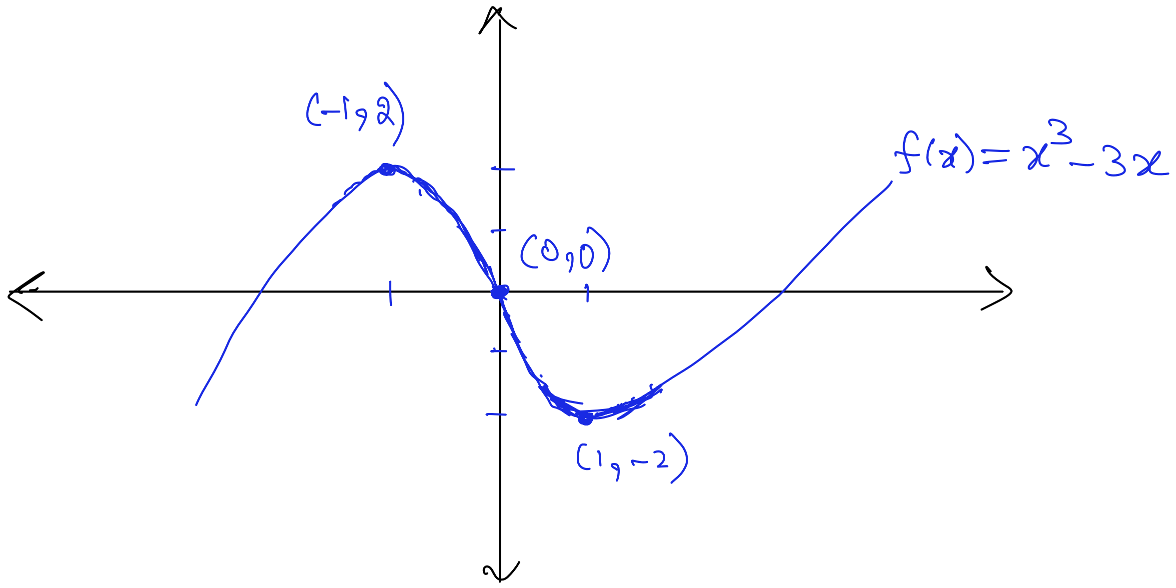
Example 3. Sketch the curve $y = x^3 - 3x$.

- minimum pt. at $(1, -2)$
- Maximum pt. at $(-1, 2)$.

$f''(x) = 6x$



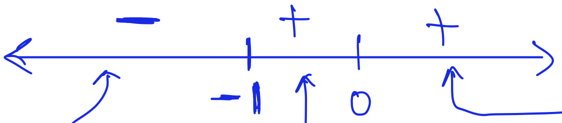
- Inflection pt at $(0, 0)$ also y-int.



Example 4. Sketch the curve $y = x^4 + (4/3)x^3$.

$$y' = 4x^3 + 4x^2 = 4x^2(x+1)$$

critical numbers: $y' = 0 \Rightarrow x^2(x+1) = 0 \Rightarrow x = 0$ or $x = -1$



$y'(-2) = 4(-2)^2(-2+1) = -16$

$y'(-\frac{1}{2}) = 4(\frac{-1}{2})^2(\frac{-1}{2}+1) = \frac{1}{2}$

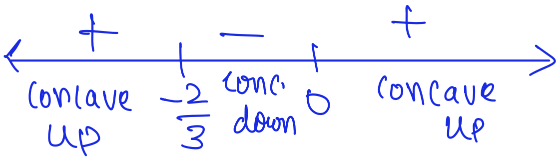
$y'(1) = 4(1)^2(1+1) = 8$

$\Rightarrow x = -1$ is a pt. of minima. $\Rightarrow f(-1) = (-1)^4 + \frac{4}{3}(-1)^3 = 1 - \frac{4}{3} = \frac{-1}{3}$

$\Rightarrow (-1, \frac{-1}{3})$ is a min. pt.

$$y'' = 12x^2 + 8x = 4x(3x+2) \Rightarrow 4x(3x+2) = 0$$

$$\Rightarrow x = 0 \text{ or } x = \frac{-2}{3}$$



$\Rightarrow (0, 0)$ and $(\frac{-2}{3}, -\frac{16}{81})$ ← Inflection pts

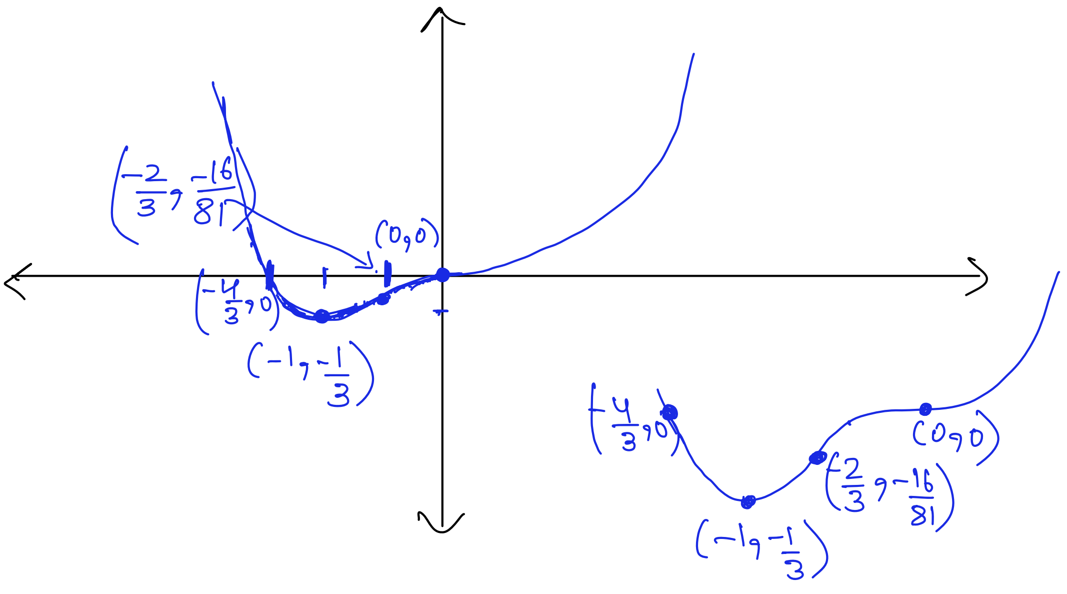
$f(\frac{-2}{3}) = (\frac{-2}{3})^4 + \frac{4}{3}(\frac{-2}{3})^3$

$= (\frac{-2}{3})^3[\frac{-2}{3} + \frac{4}{3}] = -(\frac{2}{3})^4$

x-int.

$f(x) = x^4 + \frac{4}{3}x^3 = 0$

$\Rightarrow x^3[x + \frac{4}{3}] = 0 \Rightarrow x = 0$ or $x = \frac{-4}{3} \Rightarrow (\frac{-4}{3}, 0)$



Example 5. Sketch the curve $y = \frac{x}{\sqrt{x-1}}$.

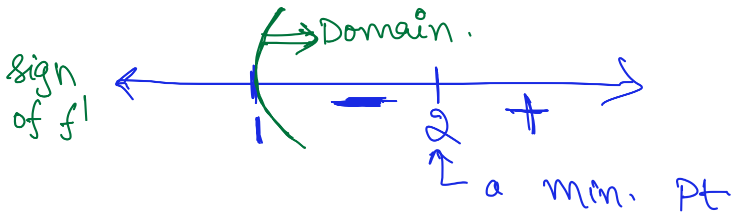
Domain = $(1, \infty)$

- Max/min Pts. : $y' = \frac{(x)'\sqrt{x-1} - x(\sqrt{x-1})'}{(\sqrt{x-1})^2}$

$$= \frac{\sqrt{x-1} - \frac{x}{2\sqrt{x-1}}}{(x-1)} = \frac{2(x-1)-x}{2\sqrt{x-1}\,(x-1)} = \frac{(x-2)}{\underbrace{2\sqrt{x-1}}_{\text{Positive}}\,\underbrace{(x-1)}_{\text{Positive}}}$$

$x > 1$

$\Rightarrow x-1 > 0$



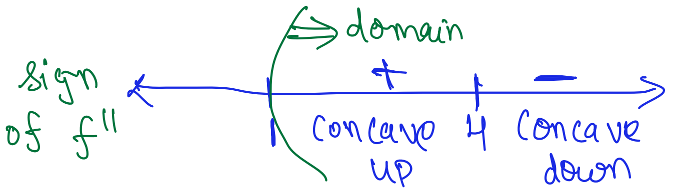
$\Rightarrow$ $(2, 2)$ is a min. pt.

$f(2) = \frac{2}{\sqrt{2-1}} = 2$

$$\frac{x-2}{2(x-1)^{3/2}}$$

- Inflection Pts. : $y'' = \frac{2(x-1)^{3/2} - (x-2)\cancel{2}\left[\frac{3}{\cancel{2}}(x-1)^{1/2}\right]}{4(x-1)^3}$

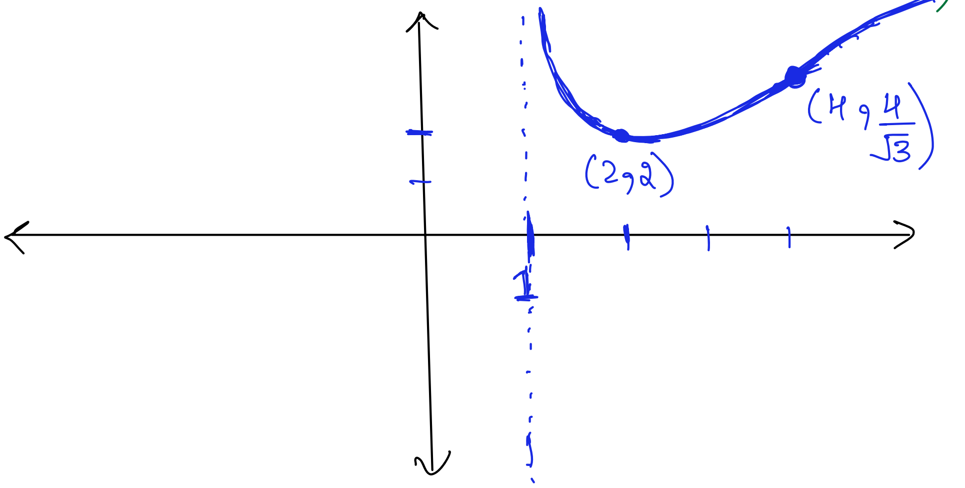
$$= \frac{(x-1)^{1/2}\left[\overbrace{2(x-1)-3(x-2)}^{-x+4}\right]}{4(x-1)^3} = \frac{-(x-1)^{1/2}(x-4)}{4(x-1)^3}$$



$\Rightarrow$ $\left(4, \frac{4}{\sqrt{3}}\right)$ is an inflection pt.

$f(4) = \frac{4}{\sqrt{4-1}}$

⊛ No intercepts ($x = 0$ is not in domain)

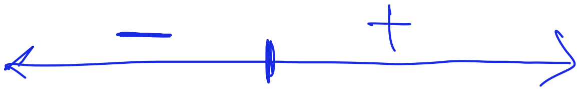


Example 6. Sketch the curve $y = (x-2)^{2/3}$.

- Min | Max Pts. $y' = \frac{2}{3}(x-2)^{\frac{2}{3}-1} = \frac{2}{3}(x-2)^{-\frac{1}{3}}$

$$= \frac{2}{3(x-2)^{1/3}} \qquad \boxed{y'(2) = \pm\infty}$$

DNE



→ min pt

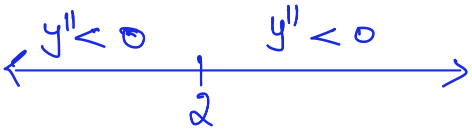
↑ $y' \to \infty$ at $x = 2$

(2, 0) is a min. Pt.

- Inflection Pts. $y'' = \frac{d}{dx}\left[\frac{2}{3}(x-2)^{-1/3}\right]$

$$= \frac{2}{3}\left(\frac{-1}{3}\right)(x-2)^{-\frac{1}{3}-1} = \frac{-2}{9}(x-2)^{-\frac{4}{3}}$$

$$= \frac{-2}{9}\left[(x-2)^{-1/3}\right]^4$$

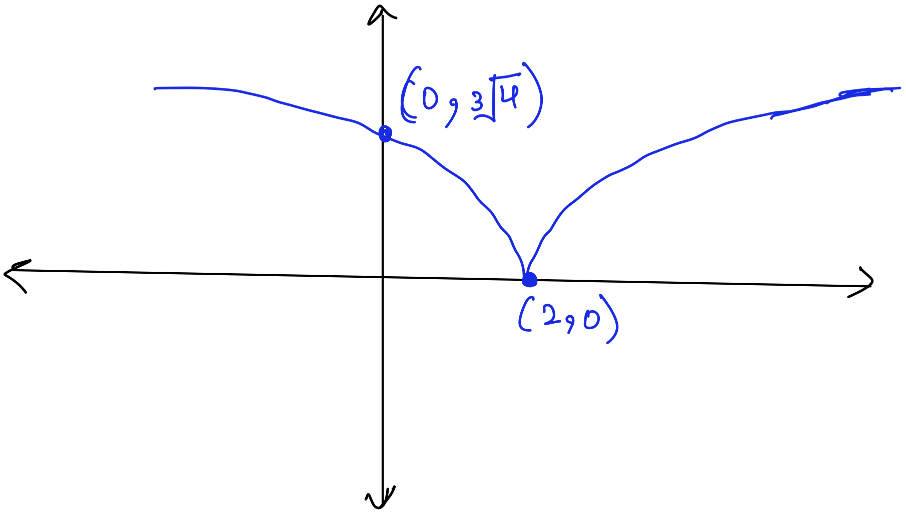


Always concave down.

Intercepts : $y = (x-2)^{2/3}$ $\xrightarrow{x\text{-int}}$ $(x-2)^{2/3} = 0 \Rightarrow x-2=0 \Rightarrow x=2$

$\xrightarrow{y\text{-int}}$ $x = 0 \Rightarrow y(0) = (0-2)^{2/3}$

$$= \left(-\sqrt[3]{2}\right)^2 = \sqrt[3]{4}$$



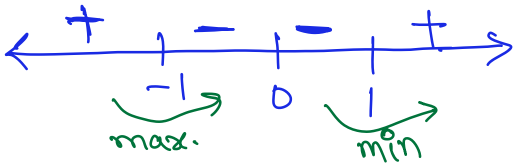
Example 7. Sketch the curve $y = x + \frac{1}{x}$.

Min|max Pts.: $y' = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2} = \frac{(x-1)(x+1)}{x^2}$

Critical Pts.: $y' = 0 \Rightarrow \frac{x^2-1}{x^2} = 0 \Rightarrow x^2 - 1 = 0$

$\Rightarrow x = \pm 1$

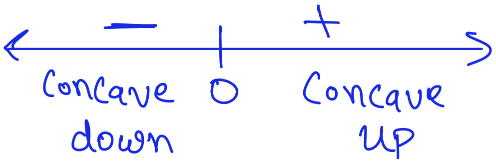
$y' = \pm\infty \Rightarrow x^2 = 0 \Rightarrow x = 0$



(1, 2) is a min pt.

(−1, −2) is a max. Pt.

Inflection Pts.: $y'' = \frac{2}{x^3}$ ⇒ y'' is never zero and $y'' = \pm\infty$ if $x = 0$



• $x = 0$ is an inflection pt. but its not in the domain.

Intercepts: No y-int since 0 not in domain.

$x + \frac{1}{x} = 0 \Rightarrow \frac{x^2+1}{x} = 0 \Rightarrow x^2 + 1 = 0$

⇒ No real solns.

⇒ No x-int.