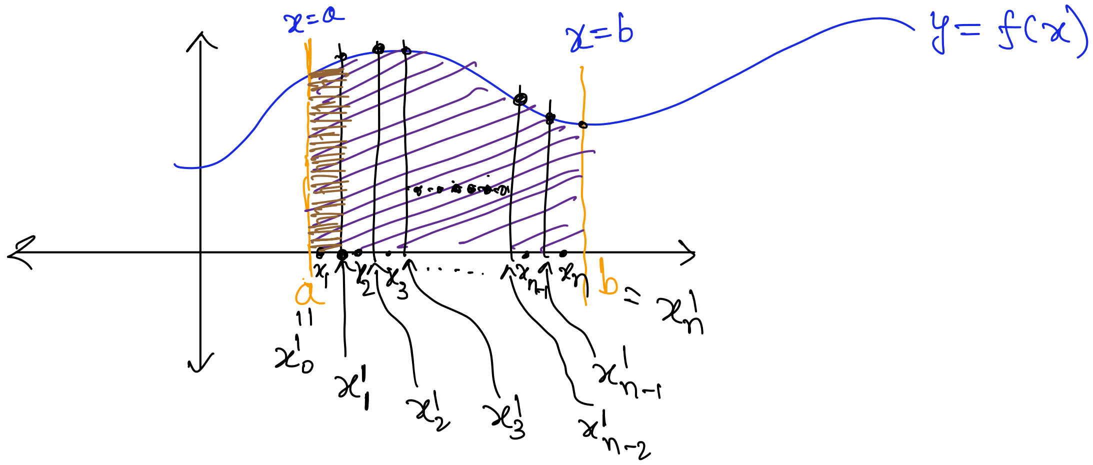
Sigma Notation

1. $\displaystyle\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

2. $\displaystyle\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

3. $\displaystyle\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$.

The Area Problem: Find the area enclosed between the curve $y = f(x)$ and the x-axis from $x = a$ to $x = b$.



$$[a, b] = \underset{\substack{\| \\ a}}{[x'_0}, x'_1] \cup [x'_1, x'_2] \cup [x'_2, x'_3] \cup \cdots\cdots \cup [x'_{n-2}, x'_{n-1}] \cup [x'_{n-1}, \underset{\substack{\| \\ b}}{x'_n}]$$

$x_1 \quad x_2 \quad x_3 \quad \cdots\cdots \quad x_{n-1} \quad x_n$

$$x'_{i-1} \le x_i \le x'_i, \quad i = 1, 2, \ldots, n-1, n$$

The i^{th} rectangular strip has width $\Delta x_i = x'_i - x'_{i-1}$ and height $f(x_i)$

Area under the curve $\simeq$ sum of the areas of rectangular strips.

$$\simeq f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \cdots\cdots + f(x_{n-1})\Delta x_{n-1} + f(x_n)\Delta x_n$$

$$\simeq \sum_{i=1}^{n} \underbrace{f(x_i)}_{\text{height of } i^{th} \text{ rectangle}} \underbrace{\Delta x_i}_{\text{width of } i^{th} \text{ rectangle.}}$$

The area is $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i$ which is denoted by $\int_a^b f(x)\,dx$ and is called the definite integral of f from a to b.

limit of a sum.

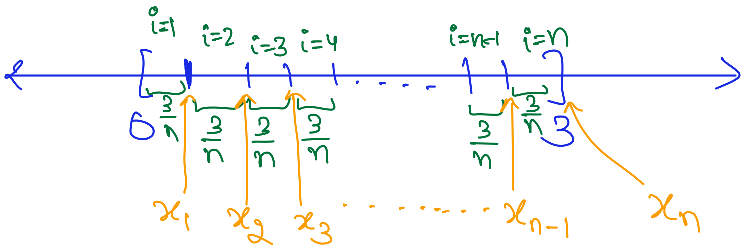
The x here is a dummy variable so we have

$$\int_a^b f(x)\,dx = \int_a^b f(y)\,dy = \int_a^b f(z)\,dz = \int_a^b f(w)\,dw.$$

Example 1. Evaluate $\int_0^3 x^2\,dx$ using the definition of definite integral.

Divide $[0, 3]$ into n subintervals of equal width.

Then, $n(\Delta x_i) = (3-0) \Rightarrow \Delta x_i = \frac{3}{n}$



$$x_1 = 0 + \frac{3}{n} = 1\left(\frac{3}{n}\right)$$

$$x_2 = 0 + \frac{3}{n} + \frac{3}{n} = 2\left(\frac{3}{n}\right)$$

$$x_3 = x_2 + \frac{3}{n} = 2\left(\frac{3}{n}\right) + \left(\frac{3}{n}\right) = 3\left(\frac{3}{n}\right)$$

$$x_4 = x_3 + \frac{3}{n} = 3\left(\frac{3}{n}\right) + \left(\frac{3}{n}\right) = 4\left(\frac{3}{n}\right)$$

$$\vdots$$

$$x_i = i\left(\frac{3}{n}\right)$$

$$x_i = x_{i-1} + \Delta x_i = x_{i-1} + \frac{3}{n}$$

$$f(x_i) = x_i^2$$

$$\sum_{i=1}^{n} f(x_i)\,\Delta x_i = \sum_{i=1}^{n} \left(i\frac{3}{n}\right)^2 \left(\frac{3}{n}\right)$$

$$= \sum_{i=1}^{n} i^2 \left(\frac{3}{n}\right)^2 \left(\frac{3}{n}\right) = \sum_{i=1}^{n} i^2 \left(\frac{3}{n}\right)^3$$

$$= \sum_{i=1}^{n} i^2 \frac{3^3}{n^3} = \sum_{i=1}^{n} i^2 \frac{27}{n^3}$$

$$= \frac{27}{n^3}(1)^2 + \frac{27}{n^3}(2)^2 + \frac{27}{n^3}(3)^2 + \cdots + \frac{27}{n^3}(n)^2$$

$$= \frac{27}{n^3} \underbrace{\sum_{i=1}^{n} i^2}_{= \frac{n(n+1)(2n+1)}{6}} = \frac{27}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} f(x_i)\,\Delta x_i = \frac{27\,n(n+1)(2n+1)}{6n^3}$$

$$\int_0^3 x^2\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i = \lim_{n\to\infty} \frac{27\cancel{n}(n+1)(2n+1)}{6n^{\cancel{3}\,2}}$$

$$= \lim_{n\to\infty} \frac{27(n+1)(2n+1)}{6n^2}$$

Divide both num. and den. by n^2

$$= \lim_{n\to\infty} \frac{27\frac{(n+1)(2n+1)}{n^2}}{6\frac{\cancel{n^2}}{\cancel{n^2}}}$$

$$= \lim_{n\to\infty} \frac{27}{6} \cdot \frac{(n+1)(2n+1)}{n^2} = \lim_{n\to\infty} \frac{27}{6}\left(\frac{n+1}{n}\right) \cdot \left(\frac{2n+1}{n}\right)$$

$$= \lim_{n\to\infty} \frac{27}{6}\left(\frac{\cancel{n}}{\cancel{n}} + \frac{1}{n}\right)\left(\frac{2\cancel{n}}{\cancel{n}} + \frac{1}{n}\right)$$

$$= \lim_{n\to\infty} \frac{27}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)$$

$$= \frac{27}{6}(1+0)(2+0) = \frac{27}{\cancel{6}\,3} \cdot \cancel{2} = \frac{\cancel{27}\,9}{\cancel{3}} = 9$$

$$\boxed{\int_0^3 x^2\,dx = 9}$$