The definite integral, by definition, is given as limit of a sum:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i \,.$$

A sum of this form is called a Riemann sum.

The Indefinite Integral, of a function f, is defined to be the antiderivatives of f:

$$\int f(x)\,dx = F(x) + c \,.$$

Properties of integral:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}\,, \quad n \neq -1 \,.$$

$$\int k\,f(x)\,dx = k\int f(x)\,dx \,.$$

$$\int \big(f(x) + g(x)\big)\,dx = \int f(x)\,dx + \int g(x)\,dx \,.$$

$$\int_{-1}^{1} (1-x^2)\,dx = \int_{-1}^{1} 1\,dx - \int_{-1}^{1} x^2\,dx$$

General power formula:

$$\int u^n\,du = \frac{u^{n+1}}{n+1}$$

$$\int \underbrace{[f(x)]}_{u=f(x)}{}^n \underbrace{f'(x)\,dx}_{du} = \frac{[f(x)]^{n+1}}{n+1} + C\,, \quad n \neq -1 \,.$$

$u = f(x)$

$\frac{du}{dx} = f'(x)$

$\Rightarrow du = f'(x)\,dx$

Example 1. Evaluate the integral $\int \underbrace{(x^2+1)}_{u}{}^{-1/2}\,\underbrace{2x\,dx}_{du}\,.$

$$u = x^2+1 \Rightarrow \frac{du}{dx} = 2x \Rightarrow du = 2x\,dx$$

$$\int (x^2+1)^{-\frac{1}{2}}\,2x\,dx = \int u^{-\frac{1}{2}}\,du = \frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$

$$= \frac{u^{1/2}}{1/2} + C = 2u^{1/2} + C$$

$$\Rightarrow \int (x^2+1)^{-\frac{1}{2}}\, 2x\, dx = \boxed{2\,(\underbrace{x^2+1}_{})^{1/2} + C}$$

Put back the Value of u.

Example 2. Integrate $\displaystyle\int x^2 \underbrace{\sqrt{x^3+1}}_{u}\, dx$.

$u = x^3+1 \Rightarrow \dfrac{du}{dx} = 3x^2 \Rightarrow du = 3x^2\, dx$

$\Rightarrow \frac{1}{3}\, du = x^2\, dx$

$$I = \int x^2 \sqrt{x^3+1}\, dx = \int \sqrt{x^3+1} \cdot x^2 \cdot dx$$

$$= \int \sqrt{u} \cdot \frac{1}{3}\, du = \frac{1}{3}\int \sqrt{u} \cdot du$$

$$= \frac{1}{3}\, \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{1}{3}\, \frac{u^{3/2}}{3/2} + C$$

$$= \frac{1}{3} \cdot \frac{2}{3} \cdot u^{3/2} + C = \frac{2}{9}\, u^{3/2} + C$$

$$= \boxed{\frac{2}{9}\,(x^3+1)^{3/2} + C}$$

Put back the Value of u.

Example 3. Integrate $\int \frac{x\,dx}{\sqrt[3]{1-x^{\cancel{3}2}}}$. $\rightarrow \int \frac{x \quad dx}{\sqrt[3]{1-x^2}}$

$$I = \int \underbrace{(1-x^2)}^{-\frac{1}{3}} \underbrace{x \cdot dx}$$

$u = 1-x^2 \Rightarrow \frac{du}{dx} = -2x \Rightarrow du = -2x \cdot dx$

$\Rightarrow \frac{1}{-2} du = x\,dx.$

$$I = \int u^{-1/3} \cdot \left(\frac{-1}{2}\right) \cdot du$$

$$= \frac{-1}{2} \int u^{\frac{-1}{3}} \cdot du = \frac{-1}{2} \cdot \frac{u^{\frac{-1}{3}+1}}{\frac{-1}{3}+1} + C$$

$$= \frac{-1}{2} \cdot \frac{u^{2/3}}{2/3} + C = -\frac{3}{4} u^{2/3} + C$$

$$= \frac{-3}{4} (1-x^2)^{2/3} + C$$

Example 4. Integrate $\int (\underbrace{x^2 - 1}_{u})^2\, dx$.

$\Rightarrow u = x^2 - 1 \Rightarrow du = 2x\,dx$

$I = \int u^2 \cdot \frac{du}{2x}$ X

$$(x^2-1)^2 = (x^2-1)(x^2-1)$$

$$= x^4 - 2x^2 + 1$$

$$I = \int (x^2-1)^2\,dx = \int (x^4 - 2x^2 + 1)\,dx$$

$$= \int x^4 \cdot dx + \int -2x^2 dx + \int 1 \cdot dx$$

$$= \int x^4\,dx - 2\int x^2\,dx + \int 1 \cdot dx$$

$$\Rightarrow I = \frac{x^{4+1}}{4+1} - 2\frac{x^{2+1}}{2+1} + \frac{x^{0+1}}{0+1} + C$$

$$\Rightarrow I = \frac{x^5}{5} - \frac{2}{3}x^3 + x + C$$

Example 5. $\int_{-\sqrt{6}}^{-1} \frac{x\,dx}{\sqrt{10-x^2}}$.

$$\Rightarrow I = \int \frac{x}{\sqrt{10-x^2}}\,dx = \int (\underbrace{10-x^2}_{u})^{-1/2} \cdot \underbrace{x \cdot dx}_{-\frac{1}{2}du}$$

$$u = 10 - x^2 \Rightarrow \frac{du}{dx} = -2x$$

$$\Rightarrow du = -2 \cdot \underline{x \cdot dx} \Rightarrow -\frac{1}{2}du = x \cdot dx$$

$$I = \int u^{-1/2} \cdot -\frac{1}{2}du = -\frac{1}{2}\int u^{-1/2}\,du$$

$$= -\frac{1}{2}\,\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$

$$= -\frac{1}{2} \cdot \frac{u^{1/2}}{1/2} + C = -\frac{1}{\cancel{2}} \cdot \cancel{2}\, u^{1/2} + C$$

$$= -u^{1/2} + C \xrightarrow{u=10-x^2} = -\sqrt{10-x^2} + C$$

Now apply limits.

$$\int_{-\sqrt{6}}^{-1} \frac{x\,dx}{\sqrt{10-x^2}} = \left(-\sqrt{10-x^2} + C\right)\Big|_{-\sqrt{6}}^{-1}$$

$$= \left[-\sqrt{10-(-1)^2}\ \boxed{+C}\right] - \left[-\sqrt{10-(-\sqrt{6})^2}\ \boxed{+C}\right]$$

can be omitted   can be omitted.

$$= \left[-\sqrt{10-1} + c \right] - \left[-\sqrt{10-6} + c \right]$$

$$= \left[-\sqrt{9} + c \right] - \left[-\sqrt{4} + c \right]$$

$$= \left[-3 + c \right] - \left[-2 + c \right]$$

$$= -3 \cancel{+c} - (-2) \cancel{-c}$$

$$= -3 + 2 = -1$$