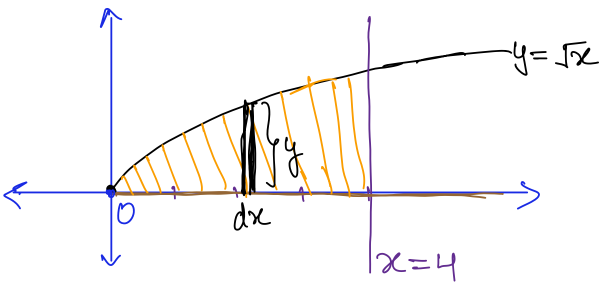
Example 1. Find area of the region bounded by $y = \sqrt{x}$, $x = 4$ and $y = 0$.



$y' = \frac{1}{2\sqrt{x}} = \frac{1}{2} x^{-1/2} > 0$ always increasing

$y'' = -\frac{1}{4} \underbrace{x^{-3/2}}_{+ve} < 0$ always concave down.

$\int_a^b \sqrt{x}\, dx \leftarrow$ Area under $y = \sqrt{x}$ from $x = a$ to $x = b$.

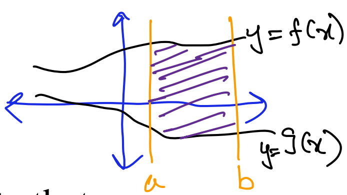
$$A = \int_0^4 \sqrt{x}\, dx$$

$$= \left. \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right|_0^4 = \left. \frac{x^{3/2}}{3/2} \right|_0^4 = \left. \frac{2}{3} x^{3/2} \right|_0^4$$

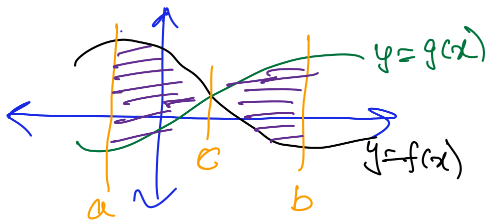
$$= \frac{2}{3} 4^{3/2} - \frac{2}{3} 0^{3/2}$$

$$= \frac{2}{3} (2)^{\cancel{2} \cdot \frac{3}{\cancel{2}}} - 0 = \frac{2}{3} \cdot 8 = \frac{16}{3}$$

Area between two curves: The area enclosed between two curves $y = f(x)$ and $y = g(x)$ from $x = a$ to $x = b$ is given by



$$A = \int_a^b \left| f(x) - g(x) \right| dx .$$



Note that

$\int_a^b (f(x) - g(x))\, dx$

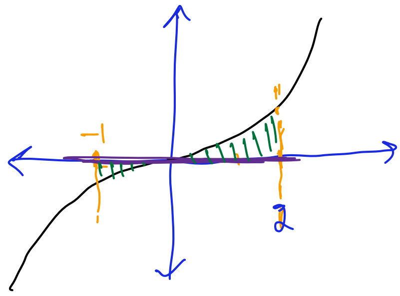
$$\left| f(x) - g(x) \right| = \begin{cases} f(x) - g(x) & \text{if } f(x) \geq g(x) \\ g(x) - f(x) & \text{if } g(x) \geq f(x) \end{cases} .$$

$\int_a^c (f(x) - g(x))\, dx + \int_c^b (g(x) - f(x))\, dx$

A property of definite integral: If f is continuous on $[a, b]$ and $a < c < b$, then

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx .$$

Example 2. Find area of the region bounded by $y = x^3$, $x = -1$, $x = 2$ and $y = 0$.



$f(x) = x^3$

$f'(x) = 3x^2 \geq 0$

Critical pt. $x = 0$

$\overset{+}{\longleftarrow}\!\!\underset{x=0}{|}\!\!\overset{+}{\longrightarrow}$

$f''(x) = 6x$

$\overset{-}{\longleftarrow}\!\!\underset{x=0}{|}\!\!\overset{+}{\longrightarrow}$

$f(x) = x^3$

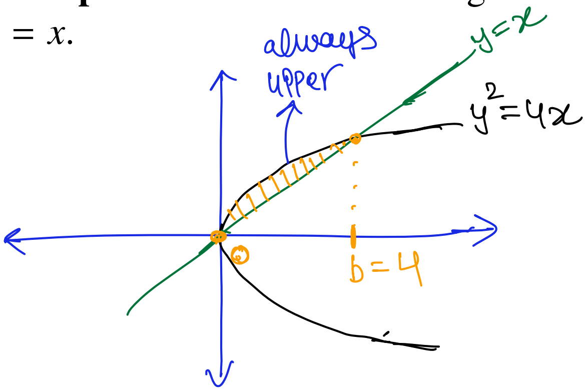
$g(x) = 0$

$$A = \int_{-1}^{2} |x^3| \, dx$$

$$= \int_{-1}^{0} (-x^3) \, dx + \int_{0}^{2} x^3 \, dx = -\frac{x^4}{4}\Big|_{-1}^{0} + \frac{x^4}{4}\Big|_{0}^{2}$$

$$= \frac{-0^4 - (-(-1)^4)}{4} + \frac{2^4 - 0^4}{4} = \frac{1}{4} + \frac{16}{4} = \frac{17}{4}$$

Example 3. Find area of the region bounded by the parabola $y^2 = 4x$ and the line $y = x$.



$y^2 = 4x$ and $y = x$

Pts. of intersection.

$\Rightarrow y^2 = 4y$

$\Rightarrow y^2 - 4y = 0 \Rightarrow y(y-4) = 0$

$\Rightarrow y = 0$ or $y = 4$

$\Rightarrow x = 0$ or $x = 4$

$y^2 = 4x \Rightarrow y = \pm\sqrt{4x}$
$= \pm 2\sqrt{x}$

We are looking for the part of $y^2 = 4x$ above the x-axis.

$\Rightarrow y = +2\sqrt{x}$

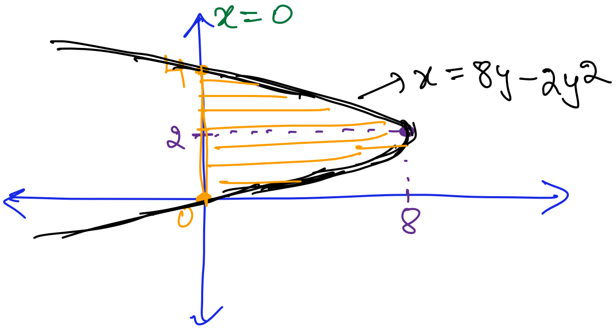
$$A = \int_0^4 \left(2\sqrt{x} - x\right) dx = 2\int_0^4 \sqrt{x} \, dx - \int_0^4 x \, dx$$

$$= 2 \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\Big|_0^4 - \frac{x^2}{2}\Big|_0^4 = \frac{4}{3}\left(4^{3/2} - 0^{3/2}\right) - \frac{4^2 - 0^2}{2} = \frac{32}{3} - 8 = \frac{8}{3}$$

Area along horizontal strips: The area bounded between the curves $x = f(y)$ and $x = g(y)$ from $y = c$ to $y = d$ is given by

$$A = \int_c^d \left| f(y) - g(y) \right| dy \,.$$

Example 4. Find the area of the region bounded by $x = 8y - 2y^2$ and y-axis.



$8y - 2y^2 = 0$

$y(8-2y) = 0$

$\Rightarrow y = 0$ or $8 - 2y = 0$

$\Rightarrow y = 4$

Parabola with axis horizontal.

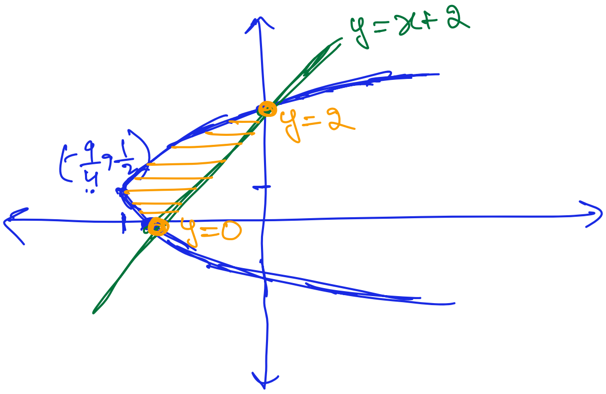
$x(2) = 8(2) - 2(2)^2 = 16 - 8 = 8$

$$A = \int_0^4 (8y - 2y^2)\, dy = 8\int_0^4 y\, dy - 2\int_0^4 y^2\, dy$$

$$= 8 \left.\frac{y^2}{2}\right|_0^4 - 2 \left.\frac{y^3}{3}\right|_0^4 = 8.\frac{(4)^2}{2} - \frac{2}{3}(4)^3 = 64 - \frac{2}{3}(64)$$

$$= \frac{64}{3}$$

Example 5. Find the area of the region bounded by $\underbrace{x = y^2 - y - 2}_{\text{Parabola}}$ and $y = x + 2$.



Vertex $\Rightarrow y = \frac{-b}{2a}$

where $x = ay^2 + by + c$

vertex $\Rightarrow y = -\frac{(-1)}{2(1)} = \frac{1}{2}$

$x = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right) - 2 = -\frac{1}{4} - 2 = -\frac{9}{4}$

⊛ Find pts of intersection of $\boxed{x = y^2 - y - 2}$, $y = x + 2$

$\Rightarrow \boxed{x = y - 2}$

$\Rightarrow y^2 - y - \cancel{2} = y - \cancel{2}$

$\Rightarrow y^2 - 2y = 0 \Rightarrow y(y-2) = 0 \Rightarrow y = 0$ or $y = 2$

$\Rightarrow x = 0 - 2 = -2$ or $x = 2 - 2 = 0$

$\int \left(\text{right curve} - \text{left curve}\right) dy$

$$\Rightarrow \quad A = \int_0^2 \left[(y-2) - (y^2 - y - 2) \right] dy$$

$$= \int_0^2 \left(y \cancel{-2} - y^2 + y \cancel{+2} \right) dy$$

$$= \int_0^2 (2y - y^2)\, dy = 2\int_0^2 y\, dy - \int_0^2 y^2\, dy$$

$$= \cancel{2} \cdot \frac{y^2}{\cancel{2}} \Big|_0^2 - \frac{y^3}{3} \Big|_0^2$$

$$= (2^2 - 0^2) - \frac{2^3 - 0^3}{3} = 4 - \frac{8}{3} = \frac{4}{3}$$