Improper integrals of type 1: When the upper or lower or both limits of the integral are infinite. In such cases we have

Recall: Fundamental Theorem of Calculus

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Type 2 ↑ if not

But f has to be continuous in $[a,b]$ and a, b have to be finite.

↓ if not Type 1

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx\,.$$

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx\,.$$

If the limit converges to some finite number we say the given improper **integral converges**, otherwise we say the given improper **integral diverges**.

When both limits are infinite we choose some point c on the real line and write the integral as a sum of two improper integrals. Then we have

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to-\infty}\int_a^c f(x)\,dx + \lim_{b\to\infty}\int_c^b f(x)\,dx\,.$$

Such an integral is said to converge if both the limits in the sum converge. Otherwise, we say the improper integral diverges.

Example 1. Evaluate the integral $\displaystyle\int_1^{\infty}\frac{dx}{x^2}$.

$$\int_1^{\infty}\frac{dx}{x^2} = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx$$

$$\int_1^b \frac{1}{x^2}\,dx = \left.\frac{x^{-2+1}}{-2+1}\right|_1^b = \left.\frac{x^{-1}}{-1}\right|_1^b = \left.\frac{-1}{x}\right|_1^b$$

$$= \frac{-1}{b} - \left(\frac{-1}{1}\right) = \frac{-1}{b} + 1$$

$$\Rightarrow \int_1^{\infty}\frac{dx}{x^2} = \lim_{b\to\infty}\left(\frac{-1}{b}+1\right) = 0 + 1 = 1$$

Example 2. Evaluate the integral $\displaystyle\int_0^\infty \frac{dx}{(x+2)^{3/2}}$.

$$\int_0^\infty \frac{dx}{(x+2)^{3/2}} = \lim_{b\to\infty} \int_0^b \frac{dx}{(x+2)^{3/2}}$$

$$\int_0^b \frac{dx}{(x+2)^{3/2}} = \int_0^b (x+2)^{-\frac{3}{2}}\, dx$$

$$\int (x+2)^{-\frac{3}{2}}\, dx = \int u^{-3/2}\, du = \frac{u^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} \quad \text{(Power rule for integration)}$$

by substituting $u = x+2$ $\Rightarrow du = dx$

$$\Rightarrow \int_0^b (x+2)^{-\frac{3}{2}} dx = \left.\frac{(x+2)^{-\frac{3}{2}+1}}{-\frac{3}{2}+1}\right|_0^b = \frac{(b+2)^{-\frac{1}{2}}}{-\frac{1}{2}} - \frac{2^{-\frac{1}{2}}}{-\frac{1}{2}} = -2\left[\frac{1}{\sqrt{b+2}} - \frac{1}{\sqrt{2}}\right]$$

$$\int_0^\infty \frac{dx}{(x+2)^{3/2}} = \lim_{b\to\infty} \frac{-2}{\sqrt{b+2}} + \frac{2}{\sqrt{2}} = 0 + \frac{2}{\sqrt{2}} = \sqrt{2}.$$

Example 3. Evaluate the integral $\displaystyle\int_{-\infty}^1 \frac{dx}{(3-x)^{5/3}}$.

$$\parallel$$

$$\lim_{a\to-\infty} \int_a^1 \frac{dx}{(3-x)^{5/3}}$$

$$\int_a^1 \frac{dx}{(3-x)^{5/3}} = \int_a^1 \underbrace{(3-x)}_{u}{}^{-5/3}\, dx$$

Let $u = 3-x \Rightarrow du = -dx \Rightarrow dx = -du$

$$\Rightarrow \int (3-x)^{-\frac{5}{3}} dx = \int u^{-\frac{5}{3}} (-du) = -\int u^{-5/3} du = -\frac{u^{-\frac{5}{3}+1}}{-\frac{5}{3}+1} + C$$

$$= -\frac{u^{-2/3}}{-2/3} + C = \frac{3}{2} u^{-2/3} + C$$

$$\Rightarrow \int_a^1 (3-x)^{-5/3} dx = \left.\frac{3}{2}(3-x)^{-\frac{2}{3}}\right|_a^1 = \frac{3}{2}(3-1)^{-2/3} - \frac{3}{2}(3-a)^{-2/3}$$

$$= \frac{3}{2}\cdot 2^{-2/3} - \frac{3}{2(3-a)^{2/3}}$$

Now take $\lim a \to -\infty$:-

$$\Rightarrow \int_{-\infty}^{1} \frac{dx}{(3-x)^{5/3}} = \lim_{a \to -\infty} \frac{3}{2 \cdot 2^{2/3}} - \frac{3}{2(3-a)^{2/3}} = \frac{3}{2^{5/3}} - \frac{3}{2 \cdot (3+\infty)^{2/3}}$$

goes to 0 since denominator goes to ∞.

$$= \boxed{\frac{3}{2^{5/3}}}$$

Improper integrals of type 2. When the integrand is discontinuous at some point in the interval of integration.

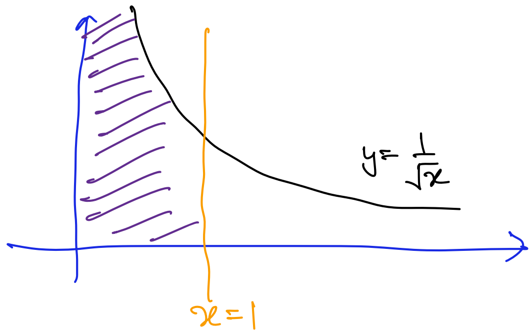
Suppose $f(x)$ is discontinuous at $x = a$. Then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx\,.$$

Suppose $f(x)$ is discontinuous at $x = b$. Then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx\,.$$

Example 4. Find the area bounded by the curve $y = 1/\sqrt{x}$, $x = 1$ and the coordinate axes.



$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\,dx$$

discontinuous at $x=0$ (Type 2)

$$\int_a^1 \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\,\Big|_a^1 = 2\sqrt{1} - 2\sqrt{a} = 2 - 2\sqrt{a}$$

Power rule for integration.

$$A = \lim_{a\to 0^+} 2 - 2\sqrt{a} = 2.$$

Example 5. Evaluate the integral $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}}$.

Type 1

$$\int_1^\infty \frac{dx}{\sqrt{x}} = \lim_{b\to\infty} \int_1^b \frac{dx}{\sqrt{x}}$$

$$\int_1^b \frac{dx}{\sqrt{x}} = \int_1^b x^{-1/2}\,dx = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\Bigg|_1^b = 2\sqrt{x}\,\Big|_1^b$$

$$= 2\sqrt{b} - 2$$

Now, $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}} = \lim_{b\to\infty} \underbrace{2\sqrt{b}}_{\to\infty} - 2 = \infty$

$\Rightarrow$ The given improper integral is divergent.

Suppose $f(x)$ is discontinuous at $x = c$ where $a < c < b$. Then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \underbrace{\lim_{t\to c^-}\int_a^t f(x)\,dx}_{} + \underbrace{\lim_{t\to c^+}\int_t^b f(x)\,dx}_{}$$

If any one of them diverges then overall integral is said to diverge.

Example 6. Evaluate the integral $\displaystyle\int_{-1}^{2} \frac{dx}{(x-1)^2}\,dx$.

Type 2

$$\int_{-1}^{2}\frac{dx}{(x-1)^2} = \left.\frac{(x-1)^{-1}}{-1}\right|_{-1}^{2}$$

$$= \left.\frac{-1}{x-1}\right|_{-1}^{2} = \frac{-1}{1} - \left(\frac{-1}{-2}\right)$$

not correct. Since f not continuous at $x=1$

$\Rightarrow$ Fundamental theorem does not apply.

$$\int_{-1}^{2}\frac{dx}{(x-1)^2} = \underbrace{\int_{-1}^{1}\frac{dx}{(x-1)^2}}_{I_1} + \underbrace{\int_{1}^{2}\frac{dx}{(x-1)^2}}_{I_2}$$

$$= \underbrace{\lim_{c\to 1^-}\int_{-1}^{c}\frac{dx}{(x-1)^2}}_{I_1} + \underbrace{\lim_{c\to 1^+}\int_{c}^{2}\frac{dx}{(x-1)^2}}_{I_2}$$

→ First evaluate I_1

$$\int_{-1}^{c} \frac{dx}{(x-1)^2} = \int_{-1}^{c} (x-1)^{-2}\, dx$$

$$\int (x-1)^{-2}\, dx = \int u^{-2}\, du = \frac{u^{-2+1}}{-2+1} = \frac{u^{-1}}{-1} = \frac{-1}{u}$$

Substituting $u = x-1 \Rightarrow du = dx$

$$= \frac{-1}{(x-1)}$$

$$\Rightarrow \int_{-1}^{c} \frac{dx}{(x-1)^2} = \left.\frac{-1}{x-1}\right|_{-1}^{c}$$

$$= \frac{-1}{c-1} - \left(\frac{-1}{-1-1}\right) = \frac{-1}{c-1} - \frac{1}{2}$$

$$\Rightarrow I_1 = \int_{-1}^{1} \frac{dx}{(x-1)^2} = \lim_{c \to 1^-} \left(\frac{-1}{c-1} - \frac{1}{2}\right)$$

$$= \frac{-1}{\to 0} - \frac{1}{2} = -\infty$$

$\Rightarrow$ I_1 diverges.

Since I_1 diverges the overall integral also diverges.

(*) There is no need to evaluate I_2. But just in case

$$\int_{c}^{2} \frac{dx}{(x-1)^2} = \left.\frac{-1}{(x-1)}\right|_{c}^{2} = \frac{-1}{(2-1)} - \frac{-1}{c-1}$$

$$= -1 + \frac{1}{c-1}$$

$$\Rightarrow I_2 = \int_{1}^{2} \frac{dx}{(x-1)^2} = \lim_{c \to 1^+} \left(-1 + \frac{1}{\to 0}\right) = +\infty$$

$\Rightarrow$ I_2 diverges.