Trapezoidal rule:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\Big(f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)\Big)$$

1, 2, 2, ..., 2, 1

where $x_i = a + \dfrac{i(b-a)}{n}$, $i = 0, 1, 2, \ldots, n$.

Example 1. Use the trapezoidal rule with $n = 3$ to approximate the integral $\displaystyle\int_{-1}^{2} \frac{2}{\sqrt{x^2+1}}\,dx$. Given that $\sqrt{5} \approx 2.236$ and $\sqrt{2} \approx 1.414$.

$n=3$, $a=-1$, $b=2$ $\Rightarrow$ $\dfrac{b-a}{2n} = \dfrac{1}{2}\left(\dfrac{2-(-1)}{3}\right) = \dfrac{1}{2}\cdot 1 = \dfrac{1}{2}$

$x_0 = a = -1$

$x_1 = -1 + \dfrac{b-a}{n} = -1+1 = 0$

$x_2 = -1 + 2\left(\dfrac{b-a}{n}\right) = -1+2 = 1$

$x_3 = -1 + 3\left(\dfrac{b-a}{n}\right) = -1+3 = 2$

$$\int_{-1}^{2} \frac{2}{\sqrt{x^2+1}}\,dx \simeq \frac{1}{2}\Big[\underset{1}{f(-1)} + \underbrace{2f(0) + 2f(1)}_{2} + \underset{1}{f(2)}\Big]$$

$$= \frac{1}{2}\left[\frac{2}{\sqrt{(-1)^2+1}} + 2\cdot\frac{2}{\sqrt{0^2+1}} + 2\cdot\frac{2}{\sqrt{1^2+1}} + \frac{2}{\sqrt{2^2+1}}\right]$$

$$= \frac{1}{2}\left[\frac{2}{\sqrt{2}} + 2\cdot\frac{2}{\sqrt{1}} + 2\cdot\frac{2}{\sqrt{2}} + \frac{2}{\sqrt{5}}\right]$$

$$= \frac{1}{2}\left[\sqrt{2} + 4 + 2\sqrt{2} + \frac{2\sqrt{5}}{5}\right]$$

$\sqrt{5} = 2.236$, $\sqrt{2} = 1.414$

Simpson's rule: 1 4 2 4 2 4 2 4 2 4 1

$$\int_a^b f(x)\,dx \approx \frac{b-a}{3n}\Big(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\Big)$$

where n is an even integer and $x_i = a + \dfrac{i(b-a)}{n}$, $i = 0, 1, 2, \ldots, n$.

Example 2. Use the Simpson's rule with $n = 4$ to approximate the integral $\displaystyle\int_{-2}^{2} \frac{dx}{x^2+1}$.

$n = 4$, $\Delta x = \dfrac{b-a}{n} = \dfrac{2-(-2)}{4} = 1$

$x_0 = -2$

$x_1 = -2 + \Delta x = -1$

$x_2 = -2 + 2(\Delta x) = 0$

$x_3 = -2 + 3(\Delta x) = 1$

$x_4 = -2 + 4(\Delta x) = 2$

$$\int_{-2}^{2} \frac{1}{x^2+1}\,dx = \frac{1}{3}\Big(f(-2) + 4f(-1) + 2f(0) + 4f(1) + f(2)\Big)$$

$$= \frac{1}{3}\left[\frac{1}{(-2)^2+1} + 4\cdot\frac{1}{(-1)^2+1} + 2\cdot\frac{1}{0^2+1} + 4\cdot\frac{1}{1^2+1} + \frac{1}{2^2+1}\right]$$

$$= \frac{1}{3}\left[\frac{1}{5} + \frac{4}{2} + \frac{2}{1} + \frac{4}{2} + \frac{1}{5}\right]$$

$$= \frac{1}{3}\left[0.2 + 2 + 2 + 2 + 0.2\right] = \frac{1}{3}(6.4)$$

$$\approx 2.13$$