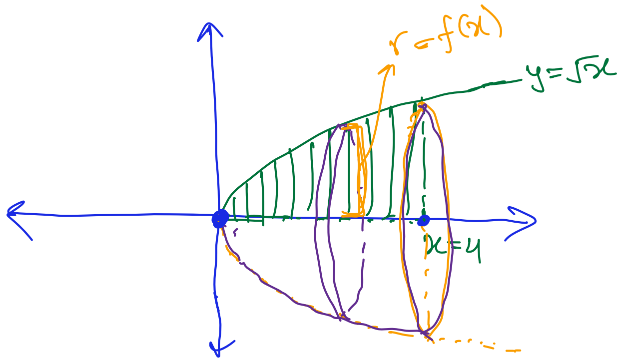
The Disk method:

Volume of disk = π . (radius)^2 . thickness = $\pi\,[f(x)]^2\,dx$

$$\text{Volume} = \pi \int_a^b [f(x)]^2\,dx\,.$$

Example 1. Find the volume of the solid obtained by revolving the region bounded by $y = \sqrt{x}$, $x = 4$ and the x-axis about the x-axis.



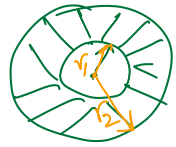
$$\text{Volume} = \pi \int_0^4 [f(x)]^2\,dx = \pi \int_0^4 (\sqrt{x})^2\,dx$$

$$= \pi \int_0^4 x\,dx = \pi \left.\frac{x^2}{2}\right|_0^4 = \pi\,\frac{(4)^2}{2}$$

$$= 8\pi$$

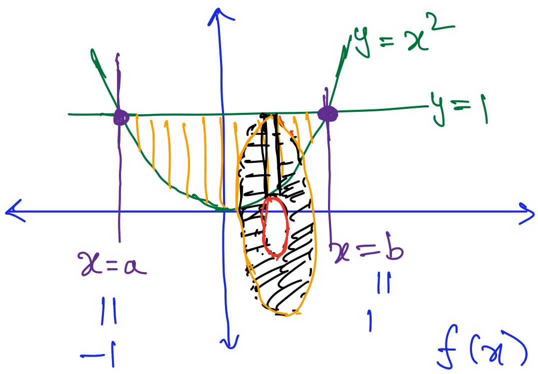
The Washer method:

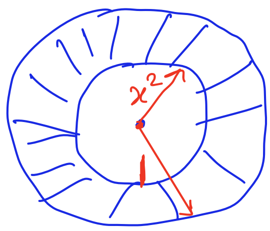
Volume of a washer $= \pi \cdot \left(\overset{r_2^2}{[f(x)]^2} - \overset{r_1^2}{[g(x)]^2} \right) \cdot dx$



$$\text{Volume} = \pi \int_a^b \left(\underbrace{[f(x)]^2}_{\text{upper curve}} - \underbrace{[g(x)]^2}_{\text{lower curve}} \right) dx$$

Example 2. Find the volume of the solid obtained by revolving the region bounded by $y = 1$ and $y = x^2$ about the x-axis.





$f(x) = 1$, (upper curve) $g(x) = x^2$ (lower curve)

$$\text{Volume} = \pi \int_a^b \left([1]^2 - [x^2]^2 \right) dx$$

To find a and b, solve $y=1$ and $y=x^2$ simultaneously.

$\Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

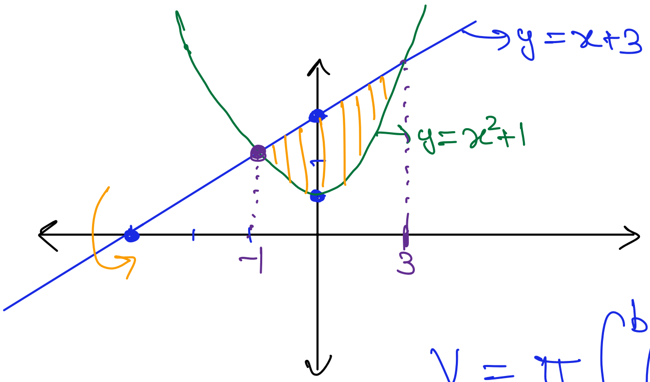
$$\text{Volume} = \pi \int_{-1}^{1} (1 - x^4)\, dx$$

$$= \pi \left(x - \frac{x^5}{5} \right) \Big|_{-1}^{1} = \pi \left[\left(1 - \frac{1}{5}\right) - \left(-1 - \frac{(-1)^5}{5}\right) \right]$$

$$= \pi \left[\frac{4}{5} - \left(-1 + \frac{1}{5}\right) \right]$$

$$= \pi \left[\frac{4}{5} - \left(\frac{-4}{5}\right) \right] = \boxed{\frac{8\pi}{5}}$$

Example 3. Find the volume of the solid obtained by revolving the region bounded by $y = x + 3$ and $y = x^2 + 1$ about the x-axis.



$$V = \pi \int_a^b \left(\underbrace{[f(x)]^2}_{x+3} - \underbrace{[g(x)]^2}_{x^2+1} \right) dx$$

$$= \pi \int_a^b \left[(x+3)^2 - (x^2+1)^2 \right] dx$$

Solve $y = x+3$ and $y = x^2+1$ simultaneously to find a, b

$$x+3 = x^2+1 \Rightarrow x^2 - x - 2 = 0$$

$$\Rightarrow x^2 - 2x + x - 2 = 0$$

$$\Rightarrow x(x-2) + 1(x-2) = 0 \Rightarrow (x-2)(x+1) = 0$$

$$\Rightarrow x = -1 \text{ or } x = 2$$

$$V = \pi \int_{-1}^{2} \left[(x+3)^2 - (x^2+1)^2 \right] dx$$

$$= \pi \int_{-1}^{2} \left[(x^2 + 6x + 9) - (x^4 + 2x^2 + 1) \right] dx$$

$$= \pi \int_{-1}^{2} \left(8 + 6x - x^2 - x^4 \right) dx$$

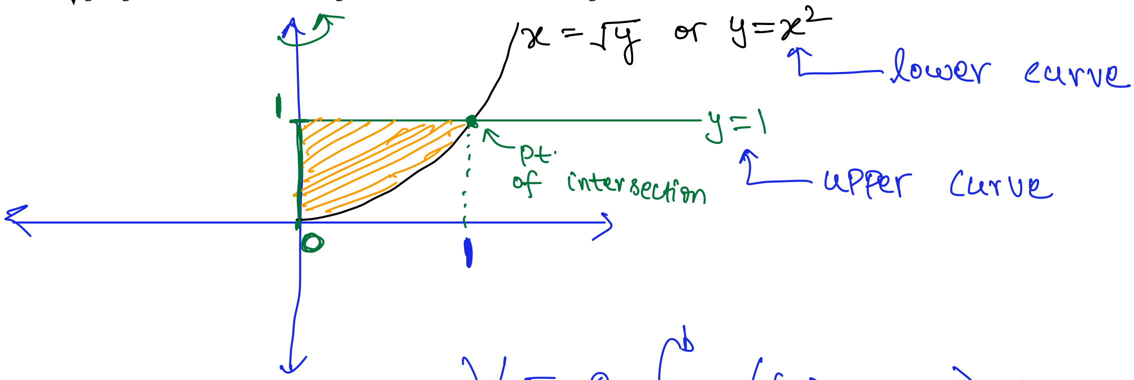
$$= \pi \left(8x + 3x^2 - \frac{x^3}{3} - \frac{x^5}{5} \right) \Bigg|_{-1}^{2}$$

The Shell method:

Volume of a shell = 2π . (radius) . (height) . (thickness) = $2\pi x\big(f(x) - g(x)\big)\,dx$

$$\text{Volume} = 2\pi \int_a^b x\big(f(x) - g(x)\big)\,dx$$

Example 4. Find the volume of the solid obtained by revolving the region bounded by $x = \sqrt{y}$, $y = 1$ and the y-axis about the y-axis.



$$V = 2\pi \int_a^b x\big(f(x) - g(x)\big)\,dx$$

$x = \sqrt{y} \Rightarrow x^2 = y$

$$= 2\pi \int_a^b x\left(1 - x^2\right)\,dx$$

$\underbrace{y = 1 \text{ and } x = \sqrt{y}}_{\text{solving simultaneously}} \Rightarrow x = \sqrt{1} = 1$

$$V = 2\pi \int_0^1 x\left(1 - x^2\right)\,dx$$

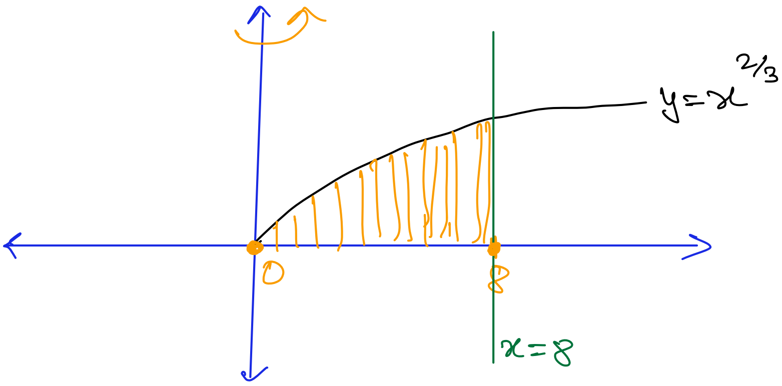
$$= 2\pi \int_0^1 \left(x - x^3\right)\,dx$$

$$= 2\pi \left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Bigg|_0^1 = 2\pi\left(\frac{1}{2} - \frac{1}{4}\right)$$

$$= \frac{2\pi}{4} = \frac{\pi}{2}$$

Example 5. Find the volume of the solid obtained by revolving the region bounded by $y = x^{2/3}$, $x = 8$ and the x-axis about the y-axis.

region.



$$V = 2\pi \int_{\underset{0}{a}}^{\overset{8}{b}} x\,(\underbrace{f(x)}_{x^{2/3}} - \underbrace{g(x)}_{0})\,dx$$

$$= 2\pi \int_0^8 x\left(x^{2/3} - 0\right) dx$$

$$= 2\pi \int_0^8 x^{1+2/3}\,dx = 2\pi \int_0^8 x^{5/3}\,dx$$

$$= 2\pi \left.\frac{x^{5/3+1}}{5/3+1}\right|_0^8 = 2\pi \cdot \frac{3}{8} \left. x^{8/3}\right|_0^8$$

$$= 2\pi \cdot \frac{3}{8} \cdot 8^{8/3}$$

$$= 2\pi \cdot \frac{3}{8} \cdot \left(2^3\right)^{8/3}$$

$$= 2\pi \cdot \frac{3}{8} \cdot 2^8 = 2\pi \cdot (3) \cdot (2^5)$$

$$= 192\pi$$