An important Limit

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 .$$

The derivatives of $y = \sin u$**,** $y = \cos u$:

$$\frac{d}{dx}(\sin x) = \cos x , \qquad \frac{d}{dx}(\sin u) = \cos u \frac{du}{dx} .$$

$$\frac{d}{dx}(\cos x) = -\sin x , \qquad \frac{d}{dx}(\cos u) = -\sin u \frac{du}{dx} .$$

Example 1. Differentiate $y = \sin \underbrace{\sqrt{x^2 + 1}}_{u}$.

$$\frac{dy}{dx} = \cos\left(\sqrt{x^2+1}\right) \cdot \frac{du}{dx}$$

$$u = \sqrt{x^2+1} = (x^2+1)^{1/2}$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{2}(x^2+1)^{\frac{1}{2}-1} \cdot \frac{d}{dx}(x^2+1)$$

(generalized power rule)

$$\Rightarrow \frac{du}{dx} = \frac{1}{\cancel{2}}(x^2+1)^{-\frac{1}{2}} \cdot \cancel{2}x$$

$$= \frac{x}{(x^2+1)^{1/2}} = \frac{x}{\sqrt{x^2+1}}$$

$$\frac{dy}{dx} = \cos\left(\sqrt{x^2+1}\right) \cdot \frac{x}{\sqrt{x^2+1}}$$

$$= \boxed{\frac{x \cos\left(\sqrt{x^2+1}\right)}{\sqrt{x^2+1}}}$$

Example 2. Find the derivative of $y = \underbrace{x^2}_{u} \underbrace{\cos x^3}_{v}$. (use Product rule)

$$(uv)' = u'v + uv'$$

$$u = x^2 \Rightarrow u' = 2x$$

$$v = \cos x^3 \Rightarrow v' = -\sin(x^3) \cdot \frac{d}{dx}(x^3)$$

$$= -\sin(x^3) \cdot 3x^2 = -3x^2 \sin(x^3)$$

$$\Rightarrow y' = u'v + uv' = 2x \cos x^3 + x^2 \left[-3x^2 \sin(x^3)\right]$$

$$= 2x \cos x^3 - 3x^4 \sin(x^3)$$

Example 3. Find the derivative of $y = \dfrac{\sin^2 x}{\sqrt{x}}$. $= \dfrac{u}{v}$ (use quotient rule)

$$y' = \frac{u'v - uv'}{v^2}$$

$$u = \sin^2 x \Rightarrow u' = \frac{d}{dx}\left[(\underbrace{\sin x}_{z})^2\right] = \frac{d}{dx}(z^2)$$

$$= \frac{d}{dz}(z^2)\,\frac{dz}{dx}$$

$$= 2(\sin x) \cos x$$

$\uparrow$ z

$$\Rightarrow u' = 2 \sin x \cos x$$

$$v = \sqrt{x}$$

$\downarrow$

$$v' = \frac{d}{dx}(x^{1/2})$$

$$= \frac{1}{2} x^{1/2 - 1}$$

$$= \frac{1}{2} x^{-1/2}$$

$$= \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$$

Therefore, $y' = \dfrac{u'v - uv'}{v^2} = \dfrac{(2 \sin x \cos x)\sqrt{x} - \sin^2 x \cdot \frac{1}{2\sqrt{x}}}{(\sqrt{x})^2}$

$$= \frac{4x \sin x \cos x - \sin^2 x}{2x\sqrt{x}}$$