The derivatives of $y = \tan u$, $y = \cot u$, $y = \sec u$, $y = \csc u$:

$$\frac{d}{dx}(\tan u) = \sec^2 u \frac{du}{dx}, \qquad \frac{d}{dx}\cot u = -\csc^2 u \frac{du}{dx}.$$

$$\frac{d}{dx}(\sec u) = \sec u \tan u \frac{du}{dx}, \qquad \frac{d}{dx}\csc u = -\csc u \cot u \frac{du}{dx}.$$

Example 1. Find the derivative of $y = \sqrt{\tan x}$.

$$y' = \frac{d}{dx}\left(\sqrt{\underbrace{\operatorname{Tan} x}_{z}}\right)$$

$$= \frac{d}{dx}\left(\sqrt{z}\right) = \frac{d}{dz}\left(\sqrt{z}\right)\frac{dz}{dx}$$

$$= \frac{1}{2\sqrt{z}}\frac{dz}{dx}$$

$$= \frac{1}{2\sqrt{\operatorname{Tan} x}}\frac{d}{dx}\left(\operatorname{Tan} x\right)$$

$$= \frac{1}{2\sqrt{\operatorname{Tan} x}}\sec^2 x$$

$$= \frac{\sec^2 x}{2\sqrt{\operatorname{Tan} x}}$$

Example 2. Find the derivative of $y = \underbrace{x}_{u}\underbrace{\sec x^2}_{v}$. (use Product rule)

$$y' = u'v + uv'$$

$$u = x \Rightarrow u' = 1$$

$$v = \sec x^2$$

$$\Rightarrow v' = \frac{d}{dx}(\sec x^2) = \sec x^2 \tan x^2 \cdot \frac{d}{dx}(x^2)$$

$$= 2x \cdot \sec(x^2) \cdot \tan(x^2)$$

$$\Rightarrow y' = \sec(x^2) + x\left[2x \sec(x^2) \cdot \tan(x^2)\right]$$

$$= \sec(x^2) + 2x^2 \cdot \sec(x^2) \cdot \tan(x^2)$$

Example 3. Differentiate $y = \underbrace{\sin 2x}_{u}\,\underbrace{\cot x^2}_{v}$. (use Product rule)

$$y' = u'v + uv'$$

$$u = \sin 2x \Rightarrow u' = \cos(2x) \cdot \frac{d}{dx}(2x)$$

$$= 2\cos(2x)$$

$$v = \cot x^2 \Rightarrow v' = -\csc^2(x^2) \cdot \frac{d}{dx}(x^2)$$

$$= -2x \csc^2(x^2)$$

$$y' = 2\cos(2x) \cdot \cot(x^2) + \sin(2x) \cdot \left[-2x \csc^2(x^2)\right]$$

$$= 2\cos(2x) \cdot \cot(x^2) - 2x \sin(2x) \csc^2(x^2)$$

Example 4. Find the derivative of $z = \sqrt{w + \csc w^3}$.

$\frac{dz}{dw}$

Let $u = w + \csc w^3$. Then use chain rule.

$$\frac{dz}{dw} = \frac{d}{dw}\left(\sqrt{u}\right) = \underbrace{\frac{d}{du}\left(\sqrt{u}\right)}_{\text{Power rule}} \frac{du}{dw} = \frac{1}{2\sqrt{u}}\,\frac{du}{dw}$$

$$\Rightarrow \frac{dz}{dw} = \frac{1}{2\sqrt{w + \csc w^3}} \cdot \frac{du}{dw}$$

Substitute u back.

Now find $\frac{du}{dw} = \frac{d}{dw}\left(w + \csc w^3\right)$

$$= 1 + \frac{d}{dw}\left(\csc w^3\right)$$

$$= 1 + \left(-\csc w^3 \cot w^3 \cdot \frac{d}{dw}\left(w^3\right)\right)$$

$$= 1 - \csc w^3 \cot w^3 \cdot \left(3w^2\right)$$

$$= 1 - 3w^2 \csc w^3 \cot w^3$$

$$\Rightarrow \frac{dz}{dw} = \frac{1}{2\sqrt{w + \csc w^3}} \cdot \left(1 - 3w^2 \csc(w^3) \cdot \cot(w^3)\right)$$

$$= \frac{1 - 3w^2 \cdot \csc(w^3) \cdot \cot(w^3)}{2\sqrt{w + \csc w^3}}$$

Example 5. Find dy/dx by implicit differentiation: $y^2 = \tan y + x$.

$$\Rightarrow (y^2)' = (\tan y)' + (x)'$$

$$(y^2)' = \frac{d}{dx}(y^2) = 2y \cdot \frac{dy}{dx}$$

$$(\tan y)' = \frac{d}{dx}(\tan y) = \sec^2 y \cdot \frac{dy}{dx}$$

$$(x)' = 1$$

$$2y \cdot \frac{dy}{dx} = \sec^2 y \cdot \frac{dy}{dx} + 1$$

$$\Rightarrow (2y - \sec^2 y)\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2y - \sec^2 y}$$