Inverse Trigonometric Functions

$$y = \arcsin x\,, \quad \text{Domain } = [-1,1]\,, \quad \text{Range } = [-\frac{\pi}{2}, \frac{\pi}{2}]\,,$$

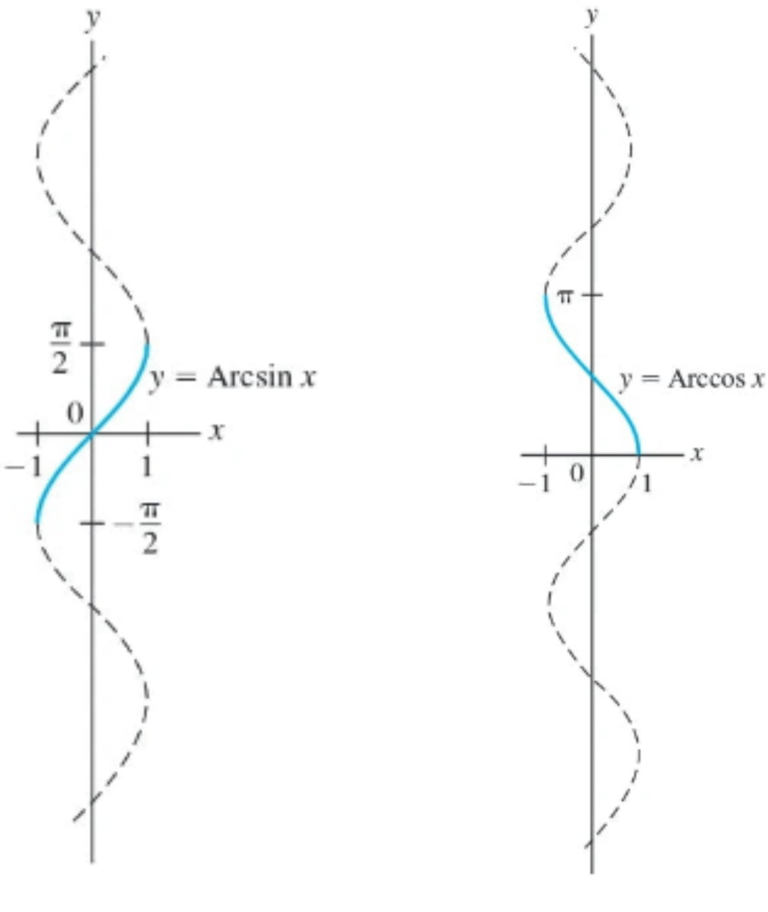
$$y = \arccos x\,, \quad \text{Domain } = [-1,1]\,, \quad \text{Range } = [0,\pi]\,,$$

$$y = \arctan x\,, \quad \text{Domain } = (-\infty,\infty)\,, \quad \text{Range } = (-\frac{\pi}{2}, \frac{\pi}{2})\,.$$

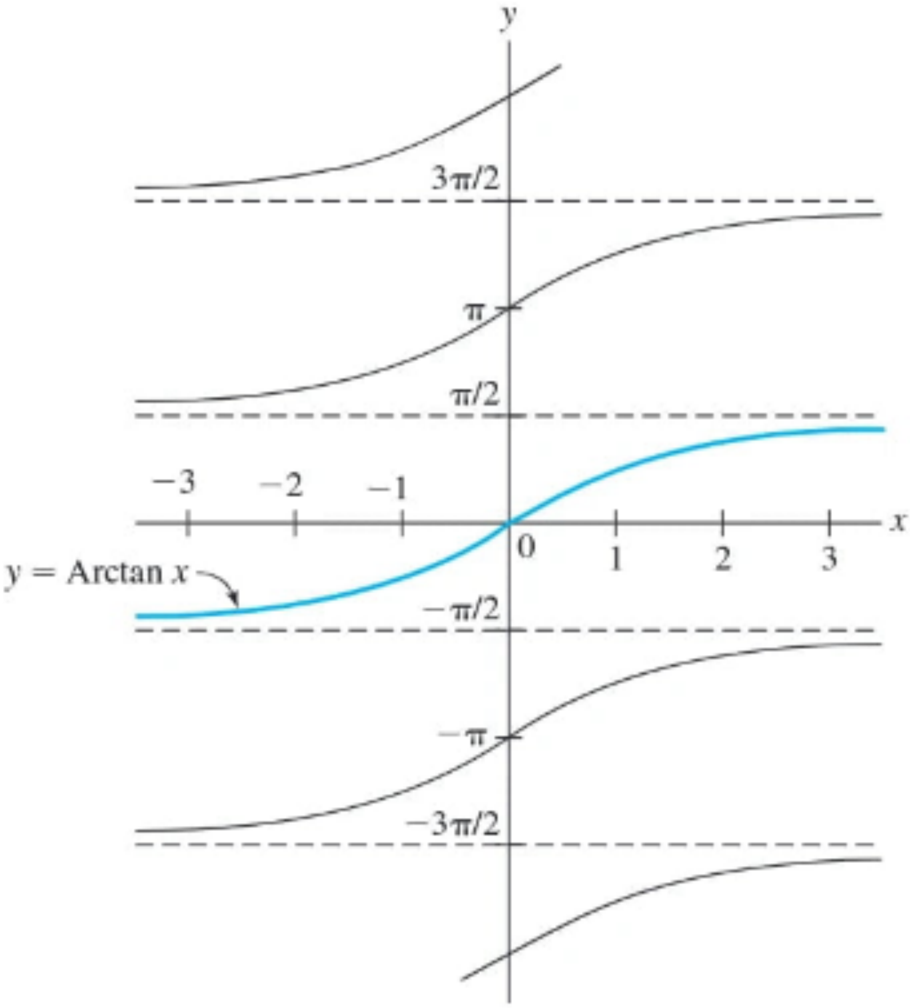
By definition,

$$\begin{aligned} y = \arcsin x &\quad \text{implies} \quad x = \sin y\,,\\ y = \arccos x &\quad \text{implies} \quad x = \cos y\,,\\ y = \arctan x &\quad \text{implies} \quad x = \tan y\,. \end{aligned}$$

Graphs of inverse trigonometric functions :



Note that $y = \arcsin x$ is an increasing functions while $y = \arccos x$ is a decreasing function. $y = \arctan x$ (shown below) is also an increasing function.



Example 1. Evaluate (a) $\arcsin\left(\frac{1}{\sqrt{2}}\right)$ (b) $\arccos\left(-\frac{\sqrt{3}}{2}\right)$ (c) $\arctan(-\sqrt{3})$.

(a) Let $\theta = \arcsin\left(\frac{1}{\sqrt{2}}\right) \Rightarrow \sin\theta = \frac{1}{\sqrt{2}}$

So, θ is the (unique) angle between $\underbrace{-\frac{\pi}{2} \text{ to } \frac{\pi}{2}}_{\text{range of arcsin}}$

whose sin ratio is $\frac{1}{\sqrt{2}}$

We know $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4} \Rightarrow \arcsin\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

(b) Let $\theta = \arccos\left(-\frac{\sqrt{3}}{2}\right) \Rightarrow \cos\theta = -\frac{\sqrt{3}}{2}$

$\Rightarrow \theta$ is (unique) angle between $\underbrace{0 \text{ to } \pi}_{\text{range of arccos}}$

whose cos ratio is $-\frac{\sqrt{3}}{2}$.

Note that Cos is negative in the second quadrant.

We know that $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \Rightarrow \cos\left(\pi - \frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

Therefore, $\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6} \Rightarrow \arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$

(c) Let $\theta = \arctan(-\sqrt{3})$. $\Rightarrow \tan\theta = -\sqrt{3}$

So, θ is the (unique) angle b/w $\underbrace{-\frac{\pi}{2} \text{ to } \frac{\pi}{2}}_{\text{range of arctan}}$ whose Tan ratio is $-\sqrt{3}$

We know that $\tan\frac{\pi}{3} = \sqrt{3} \Rightarrow \tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}$

$\Rightarrow \theta = \arctan(-\sqrt{3}) = -\frac{\pi}{3}$

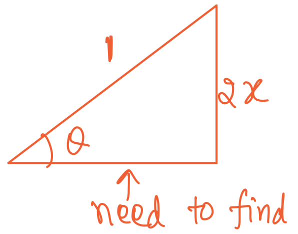
Example 2. Find an algebraic expression for tan(arcsin $2x$).

Let $\theta = \arcsin(2x)$. We want to find $\tan\theta$.

$\Downarrow$

By definition, $\sin\theta = 2x = \frac{2x}{1}$

We know $\sin\theta = \frac{P}{H} = \frac{2x}{1}$



Let $P = 2x$. Then $H = 1$

By Pythagoras theorem, $P^2 + B^2 = H^2$

$\Rightarrow (2x)^2 + B^2 = (1)^2 \Rightarrow 4x^2 + B^2 = 1$

$\Rightarrow B = \pm\sqrt{1-4x^2}$

We reject $-\sqrt{1-4x^2}$ because in the range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, both Sin and Tan are either +ve or both are −ve. This is determined by sign of x.

$\Rightarrow \tan\theta = \frac{P}{B} = \frac{2x}{\sqrt{1-4x^2}}$

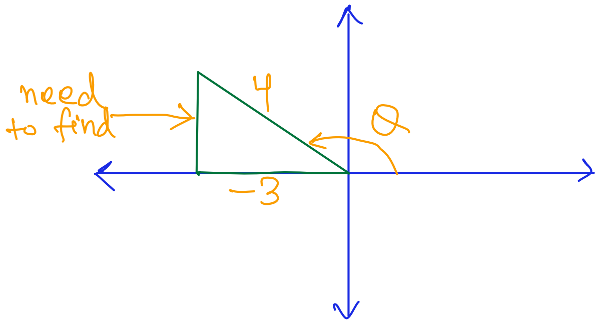
Example 3. Evaluate sin (arccos(−3/4)).

Let $\theta = \arccos\left(-\frac{3}{4}\right) \Rightarrow \cos\theta = -\frac{3}{4}$

and $\underbrace{0 \le \theta \le \pi}_{\text{range of } \arccos}$

Since $\cos\theta$ is negative in 2^{nd} quadrant

we must have $\frac{\pi}{2} \le \theta \le \pi$



$\cos\theta = \frac{-3}{4} = \frac{B}{H}$

Let $B = -3$, then $H = 4$

$P^2 + B^2 = H^2$

$\Rightarrow P^2 + (-3)^2 = (4)^2$

$\Rightarrow P^2 + 9 = 16 \Rightarrow P^2 = 7$

$\Rightarrow P = \pm\sqrt{7}$

From the figure, it is clear that P is +ve in the 2^{nd} quadrant. So we reject $-\sqrt{7}$.

$\Rightarrow P = \sqrt{7} \Rightarrow \sin\theta = \frac{P}{H} = \frac{\sqrt{7}}{4}$

$$\Rightarrow \sin\left(\arccos\left(-\frac{3}{4}\right)\right) = \frac{\sqrt{7}}{4}$$