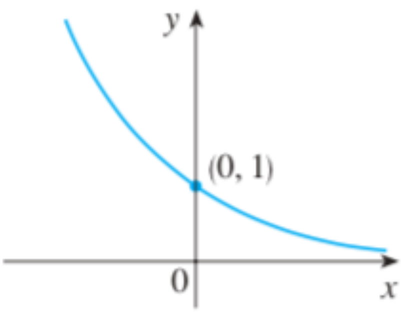
Exponential functions

The exponential functions $f(x) = b^x$ are defined for $0 < b < 1$ or $b > 1$.
The (constant) number b here is the base.

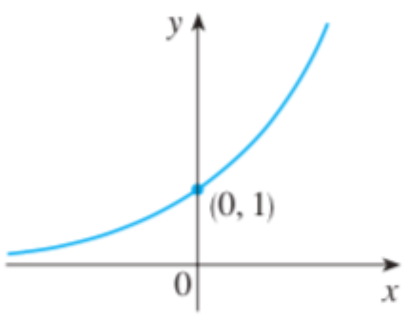
- If $x = n$, a positive integer number, then $b^n = \underbrace{b.b.\cdots b.b}_{n \text{ factors}}$.
- $b^{-n} = \dfrac{1}{b^n}$.
- If $x = 0$, then $b^0 = 1$.
- If x is a rational number then $b^x = b^{n/d} = \sqrt[d]{b^n}$.
- If x is an irrational number, we make a sequence of rational numbers r_n converging to x, and then b^{r_n} converges to b^x.

The domain of $f(x) = b^x$ is $\mathbb{R}$ and the range is $(0, \infty)$.

The graph of $f(x) = b^x$ depends on whether the base is less than 1 or greater than 1.



(a) $y = b^x$, $0 < b < 1$



(c) $y = b^x$, $b > 1$

In the first case, it is a decreasing function, while in the second case, it is an increasing function.

Properties of exponential functions

1. $b^x \,.\, b^y = b^{x+y}$, $\qquad \dfrac{b^x}{b^y} = b^{x-y}$.
2. $(b^x)^y = b^{xy}$, $\qquad (ab)^x = a^x\, b^x$.
3. If $0 < b < 1$, then $\lim\limits_{x \to -\infty} b^x = \infty$ and $\lim\limits_{x \to \infty} b^x = 0$.
4. If $b > 1$, then $\lim\limits_{x \to -\infty} b^x = 0$ and $\lim\limits_{x \to \infty} b^x = \infty$.

The natural exponential function is defined to be $f(x) = e^x$, where e (called Euler's number) is an irrational number. It's approximate value to 10 decimal places is $e \approx 2.7182818285$. In particular, $e > 1$. Sometimes e is also defined as the following limit

$$e = \lim_{h\to 0}(1+h)^{1/h} .$$

Example 1. Evaluate the limit $\lim_{x\to\infty}(2^{-x} - 1)$.

$$\lim_{x\to\infty}(2^{-x}-1) = \lim_{x\to\infty} 2^{-x} - 1 = \lim_{x\to\infty} \frac{1}{2^x} - 1$$

$$= \frac{1}{\infty} - 1 = 0 - 1 = -1$$

Logarithmic Functions

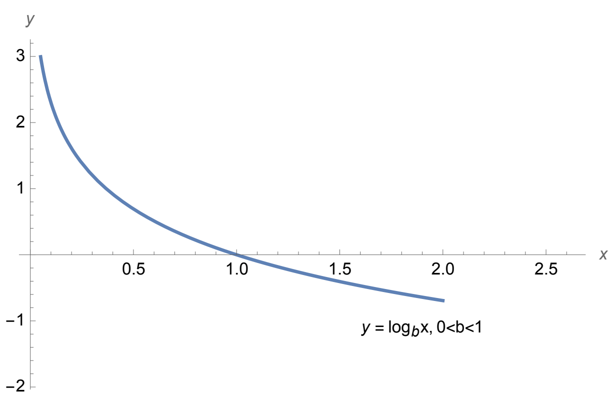
The logarithm to the base b of a positive real number x, where $b > 0$, $b \neq 1$, is written as $\log_b x$, and defined as

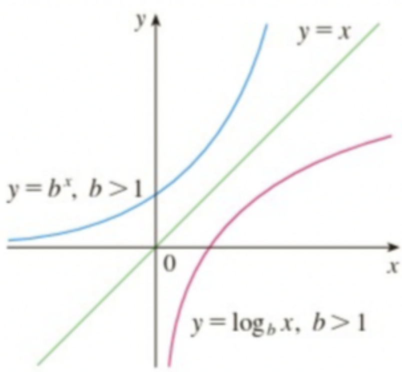
$$y = \log_b x \text{ if and only if } x = b^y .$$

The logarithmic function is defined as $f(x) = \log_b x$ where $0 < b < 1$ or $b > 1$.

The domain of $f(x) = \log_b x$ is $(0, \infty)$ and the range is $\mathbb{R}$.

The graph of $f(x) = \log_b x$ depends on whether the base b is less than 1 or greater than 1.





In the first case, it is a decreasing function, while in the second case, it is an increasing function.

Properties of logarithm

1. $\log_b(MN) = \log_b M + \log_b N$.
2. $\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$.
3. $\log_b M^k = k \log_b M$.
4. $\log_b 1 = 0$.
5. *Cancellation equations*:

$$\log_b(b^x) = x \quad \text{for every } x \in \mathbb{R}\,,$$
$$b^{\log_b x} = x \quad \text{for every } x > 0\,.$$

6. If $0 < b < 1$, then $\lim_{x\to 0^+} \log_b x = \infty$ and $\lim_{x\to\infty} \log_b x = -\infty$.
7. If $b > 1$, then $\lim_{x\to 0^+} \log_b x = -\infty$ and $\lim_{x\to\infty} \log_b x = \infty$.

The natural logarithm function is defined as $f(x) = \ln x = \log_e x$. It has the following important properties.

1. $\ln 1 = 0$ and $\ln e = 1$.
2. $\ln(e^x) = x$ and $e^{\ln x} = x$.
3. *Change of base formula*: $\log_b x = \dfrac{\ln x}{\ln b}$.

Example 2. Expand $\ln \sqrt{\dfrac{x+1}{x^2 y}}$.

$$\ln\sqrt{\frac{x+1}{x^2y}} = \ln\left(\frac{x+1}{x^2y}\right)^{1/2}$$
$$= \frac{1}{2}\ln\left(\frac{x+1}{x^2y}\right)$$
$$= \frac{1}{2}\left[\ln(x+1) - \ln(x^2y)\right]$$
$$= \frac{1}{2}\ln(x+1) - \frac{1}{2}\ln(x^2y) = \frac{1}{2}\ln(x+1) - \frac{1}{2}\left[\ln(x^2) + \ln y\right]$$
$$= \frac{1}{2}\ln(x+1) - \frac{1}{2}\ln(x^2) - \frac{1}{2}\ln y$$
$$= \frac{1}{2}\ln(x+1) - \frac{1}{2}\cdot 2\ln x - \frac{1}{2}\ln y = \frac{1}{2}\ln(x+1) - \ln x - \frac{1}{2}\ln y.$$

Example 3. Express $\ln a + \frac{1}{5}\ln b - \ln(a+b)$ as a single logarithm.

$k \ln M = \ln M^k \quad [M = b,\ k = \frac{1}{5}]$

$$\ln a + \frac{1}{5}\ln b - \ln(a+b)$$
$$= \ln a + \ln b^{1/5} - \ln(a+b)$$
$$= \ln(a \cdot b^{1/5}) - \ln(a+b)$$
$$= \ln\left(\frac{a \cdot b^{1/5}}{a+b}\right) = \ln\left(\frac{a\sqrt[5]{b}}{a+b}\right)$$

Example 4. Solve the equation $10^{5-3x} + 4 = 104$.

$$10^{5-3x} + 4 = 104$$
$$\Rightarrow 10^{5-3x} = 104 - 4$$
$$\Rightarrow 10^{5-3x} = 100$$

Take $\log_{10}$ on both sides :—

$$\Rightarrow \log_{10}(10^{5-3x}) = \log_{10} 100$$
$$\Rightarrow (5-3x)\log_{10} 10 = \log_{10} 10^2$$
$$\Rightarrow (5-3x) \cdot 1 = 2$$
$$\Rightarrow 5 - 3x = 2 \Rightarrow -3x = 2 - 5$$
$$\Rightarrow -3x = -3 \Rightarrow x = 1$$