Derivatives of logarithmic functions

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}\,.$$

$$\frac{d}{dx}(\log_b u) = \frac{1}{u\ln b}\frac{du}{dx}\,.$$

Example 1. Differentiate $y = \log_2 x^2$ with respect to x .

$$y' = [\log_2 x^2]' = \frac{1}{x^2 \ln 2}\cdot [x^2]'$$

$b=2$ $\quad u=x^2$

$$= \frac{1}{\cancel{x^2}\ln 2}\cdot 2\cancel{x} = \boxed{\frac{2}{x\ln 2}}$$

Example 2. Differentiate $T = \log_{10}(v^2+v)$ with respect to v.

$$\frac{dT}{dv} = \frac{d}{dv}\left(\log_{10}(v^2+v)\right)$$

Chain rule.

$$= \frac{1}{(v^2+v)\ln 10}\cdot\frac{d}{dv}(v^2+v)$$

$$= \frac{1}{(v^2+v)\ln 10}\cdot(2v+1)$$

$$= \boxed{\frac{2v+1}{(v^2+v)\ln 10}}$$

Example 3. Differentiate $y = \ln \sec x$ with respect to x.

$$y' = [\ln(\sec x)]' = \frac{1}{\sec x} \cdot [\sec x]'$$

$$= \frac{\cancel{\sec x} \tan x}{\cancel{\sec x}}$$

$$= \tan x$$

Example 4. Find dy/dx if $y = \ln \sqrt[3]{x^2 + 1}$.

$$\boxed{\frac{1}{\sqrt[3]{x^2+1}} \cdot \left[\sqrt[3]{x^2+1}\right]'}$$

more complicated.

Use properties of ln

$$y = \ln \sqrt[3]{x^2+1}$$

$$= \ln \underbrace{(x^2+1)}_{M}^{1/3} = \frac{1}{3} \ln(x^2+1)$$

K

$$(\ln M^K = K \ln M)$$

$$y' = \frac{1}{3}[\ln(x^2+1)]' = \frac{1}{3} \frac{1}{x^2+1} \cdot (x^2+1)'$$

$$= \frac{1}{3} \frac{2x}{x^2+1} = \frac{2x}{3(x^2+1)}$$

Example 5. Find the derivative of $y = \ln(\sin^2 x/x)$.

$\underbrace{\ln(\sin^2 x/x)}$ → do not diff. directly.

$$y = \ln\left(\frac{\sin^2 x}{x}\right)$$

$$= \ln(\sin^2 x) - \ln x$$

$$= \ln(\sin x)^2 - \ln x$$

$$= 2\ln(\sin x) - \ln x$$

$$y' = 2\left[\ln(\sin x)\right]' - \left[\ln x\right]'$$

$$= 2\,\frac{1}{\sin x}\cdot\left[\sin x\right]' - \frac{1}{x}$$

$$= 2\,\frac{\cos x}{\sin x} - \frac{1}{x} = 2\cot x - \frac{1}{x}$$

Logarithmic Differentiation: Differentiate $y = [f(x)]^{g(x)}$.

1. *Step 1*: Take ln on both sides so that $\ln y = g(x)\ln f(x)$.
2. *Step 2*: Simplify the RHS if possible.
3. *Step 3*: Differentiate both sides with respect to x.

   Note that the LHS always differentiates to $\dfrac{1}{y}\dfrac{dy}{dx}$.
4. *Step 4*: Multiply both sides with y to obtain $\dfrac{dy}{dx}$.

Example 6. Differentiate $y = x^x$.

Step 1 : $\ln y = \ln x^x = x \ln x$

Step 2 : No more simplification possible.

Step 3 : $\frac{1}{y}\frac{dy}{dx} = [x \ln x]'$

Product rule

$= [x]' \ln x + x [\ln x]'$

$= 1 \cdot \ln x + \cancel{x} \cdot \frac{1}{\cancel{x}} = \ln x + 1$

Step 4 $\frac{dy}{dx} = y(\ln x + 1) = x^x(\ln x + 1)$

Example 7. Differentiate $y = (\sin x)^{\cos x}$.

Step 1 $\ln y = \ln (\sin x)^{\cos x} = \cos x \ln(\sin x)$

Step 2 No simplification Possible.

Step 3 $\frac{1}{y}\frac{dy}{dx} = [\underbrace{\cos x}_{u} \cdot \underbrace{\ln(\sin x)}_{v}]'$

(Product rule)

$= (\cos x)' \ln(\sin x) + \cos x \cdot [\ln(\sin x)]'$

$= -\sin x \cdot \ln(\sin x) + \cos x \cdot \left[\frac{1}{\sin x} \cdot \underbrace{[\sin x]'}_{\cos x}\right]$

$= -\sin x \cdot \ln(\sin x) + \cos x \cdot \cot x$

$\cot x \to \frac{\cos x}{\sin x}$

Step 4

$$\frac{dy}{dx} = y\left[-\sin x \cdot \ln(\sin x) + \cos x \cdot \cot x\right]$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^{\cos x}\left[-\sin x \cdot \ln(\sin x) + \cos x \cdot \cot x\right]$$

HW:

$$(x)^{\sin x} \quad , \quad (\tan x)^{x}$$