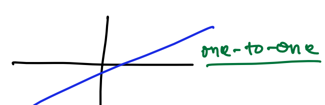
Math 110- 9.1 Notes

Given $f(x) = 3x + 2$ and $g(x) = x^2 - 4$, Find $(f \circ g)(2)$ $(f \circ g)(2) = f(g(2)) = f(0)$ $g(2) = 2^2 - 4 = 4 - 4 = 0$ $(f \circ g)(2) = f(0) = 3(0) + 2$ $(f \circ g)(2) = 2$	Given $f(x) = -2x + 1$ and $g(x) = x^3$, Find $(f \circ g)(-1)$ $(f \circ g)(-1) = f(g(-1)) = f(-1)$ $g(-1) = (-1)^3 = -1$ $(f \circ g)(-1) = f(-1) = -2(-1) + 1$ $= 2 + 1$ $\Rightarrow (f \circ g)(-1) = 3$
Given $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$, Find $(f \circ g)(x)$ $(f \circ g)(x) = f(g(x))$ $= f(x^2 + 1)$ $= \sqrt{x^2 + 1}$	Given $f(x) = \sqrt{x + 4}$ and $g(x) = \frac{5}{x}$ Find $(f \circ g)(x)$ $(f \circ g)(x) = f(g(x))$ $= f\left(\frac{5}{x}\right)$ $= \sqrt{\frac{5}{x} + 4}$



Inverses and One-to-One Functions

ESSENTIALS

One-to-One Function

A function *f* is *one-to-one* if different inputs have different outputs. For every one-to-one function, an inverse function exists.

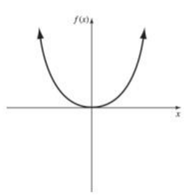
The Horizontal-Line Test

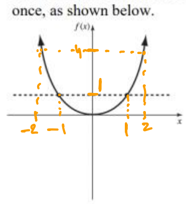
If it is impossible to draw a horizontal line that intersects a function's graph more than once, then the function is one-to-one.

Example

- Determine if the function at right is one-to-one.

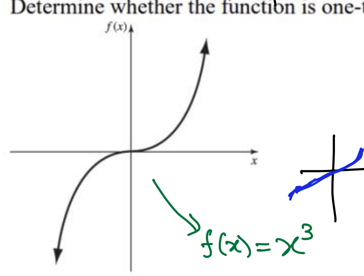
  The function is not one-to-one because there is a horizontal line that crosses the graph more than once, as shown below.





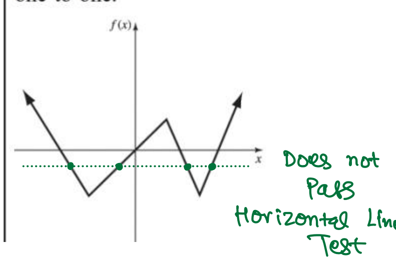
$2 \to 4$
$-2 \to 4$
$2 \leftarrow 4$
$-2 \leftarrow 4$

Determine whether the function is one-to-one.



Yes (one-to-one)

Determine whether the function is one-to-one.



Not a one-to-one function

Many-to-one

f

One-to-one

Inverse to f

Functions

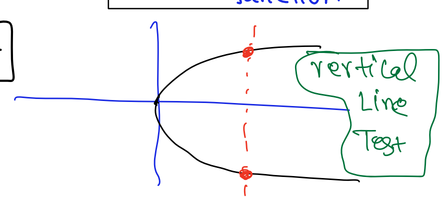
$1 \leftarrow 2$
$4 \leftarrow -3$
6

Not

Many-to-one: $1 \to 2$, $4 \to -3$, $6 \to -3$, 8

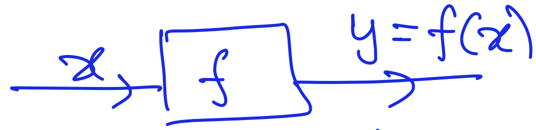
One-to-one: $1 \to 2$, $4 \to 8$, $6 \to -3$

Inverse to f: $1 \leftarrow 2$, $4 \leftarrow 8$, $6 \leftarrow -3$

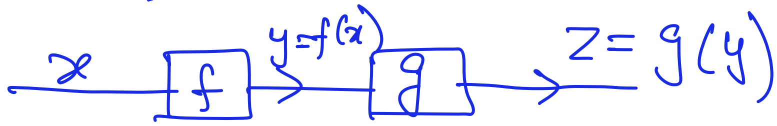


a function

Functions

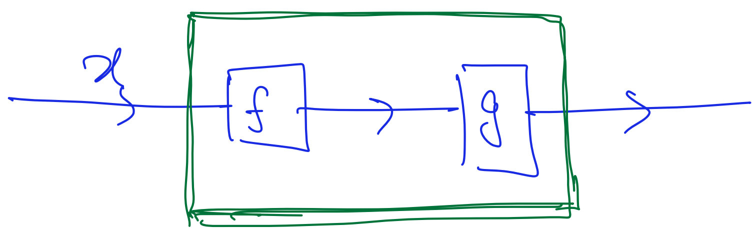


Composition



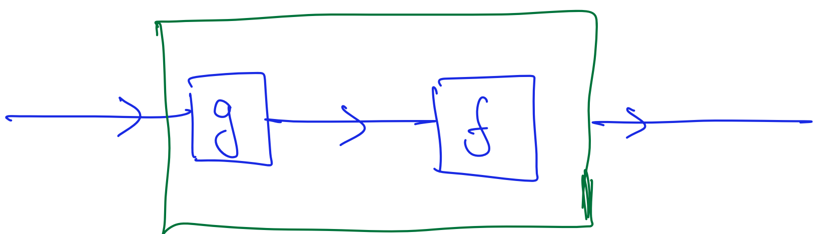
$$\Rightarrow \quad z = g(y) = g\left(f(x)\right)$$

$$= (g \circ f)(x)$$

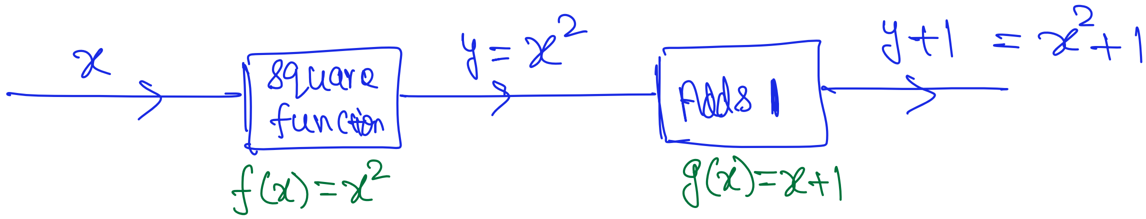


$g \circ f$

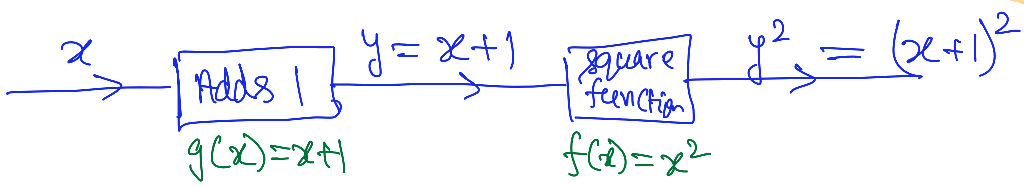
$$\boxed{g \circ f \neq f \circ g}$$



$f \circ g$



$$(g \circ f)(x) = g\left(f(x)\right) = g\left(x^2\right) = x^2 + 1 = (g \circ f)(x)$$



$$(f \circ g)(x) = f\left(g(x)\right) = f(x+1) = (x+1)^2 = (f \circ g)(x)$$

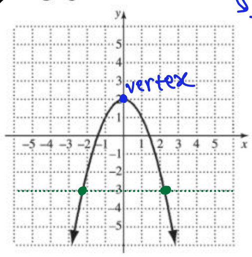
$\boxed{\text{Determine whether the function } f(x) = -x^2 + 2 \text{ is one-to-one.}}$

$\rightarrow$ We graph the function.

$-(x-0)^2+2$

$h=0$

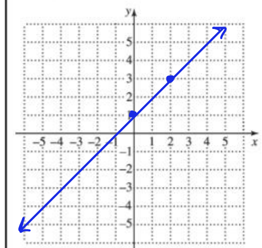
$k=2$



Does not Pass the Horizontal Line Test

$\boxed{\text{Not one-to-one}}$

Determine whether the function $f(x) = x + 1$ is one-to-one.



Passes the Horizontal Line Test

$\boxed{\text{One-to-One}}$

$f(x) = ax^3 + b$ are one-~~to-one~~

Determine whether the function $f(x) = \frac{1}{2}x + 3$ is one to one and if so, find a formula for $f^{-1}(x)$	Determine whether the function $f(x) = (x-1)^3$ is one to one and if so, find a formula for $f^{-1}(x)$	Determine whether the function ~~$f(x) = (x-1)^3$~~ is one to one and if so, find a formula for $f^{-1}(x)$
$\rightarrow$ Plot the graph and use HLT to see its one-to-one OR Linear functions are One-to-one Step 1 $y = \frac{1}{2}x + 3$ Step 2 $x = \frac{1}{2}y + 3$ Step 3 $\frac{1}{2}y + 3 = x$ $\Rightarrow \frac{1}{2}y = x - 3 \Rightarrow y = 2(x-3)$ $= 2x - 6$ Step 4 $f^{-1}(x) = 2x - 6$	$\rightarrow f(x) = (ax+b)^3$ are one-to-one (cubic functions) Step 1 $y = (x-1)^3$ Step 2 $x = (y-1)^3$ Step 3 $(y-1)^3 = x$ Take cube-root on both sides: $\sqrt[3]{(y-1)^3} = \sqrt[3]{x}$ $\Rightarrow y - 1 = \sqrt[3]{x} \Rightarrow y = \sqrt[3]{x} + 1$	$f(x) = 2(x-3)^3$ HW!

$\boxed{f^{-1}(x) \neq \frac{1}{f(x)}}$

Step 4 $f^{-1}(x) = \sqrt[3]{x} + 1$

Graphing Functions and Their Inverses

ESSENTIALS

Visualizing Inverses

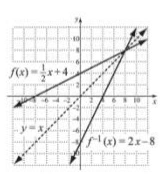
The graph of f^{-1} is a reflection of the graph of f across the line $y = x$.

Example

- Graph the function $f(x) = \frac{1}{2}x + 4$ and its inverse on the same set of axes.

First, find the inverse function. $f(x) = \cdots$

1. Replace $f(x)$ with y: $y = \frac{1}{2}x + 4$
2. Interchange x and y: $x = \frac{1}{2}y + 4$
3. Solve for y: $x - 4 = \frac{1}{2}y$
   $2(x-4) = y$
   $2x - 8 = y$
4. Replace y with $f^{-1}(x)$: $f^{-1}(x) = 2x - 8$



Notice that the graph of $f^{-1}(x)$ is the reflection of the graph of $f(x)$ across the line $y = x$.

Graph the following functions and their inverses

Graphs of $y = f(x)$ and $y = f^{-1}(x)$ are related to each other:

They are mirror images

of each other through $y=x$ line

Given f(x) show that that stated $f^{-1}(x)$ is the inverse of the function.

$f(x) = \frac{1}{4}x^3$ HW.

$f^{-1}(x) = \sqrt[3]{4x}$

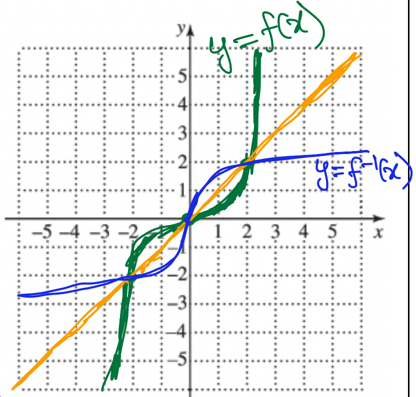
$y = \frac{1}{4}x^3$

Inter. x and y

$x = \frac{1}{4}y^3$

$4x = y^3$

$\Rightarrow y = \sqrt[3]{4x}$



$f(x) = -3x + 4$ $\quad y=-3x+4$

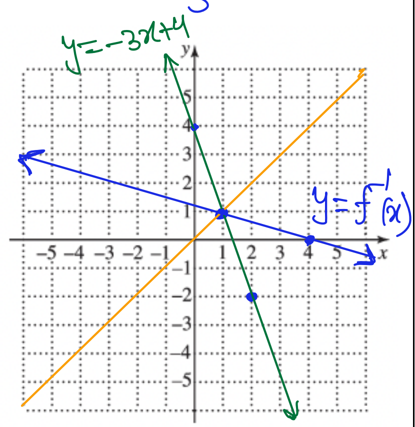
$\boxed{f^{-1}(x) = \frac{1}{3}(4-x) = -\frac{1}{3}x + \frac{4}{3}}$

$y = -3x+4$

$x = -3y+4$

$3y = 4-x$

$y = \frac{1}{3}(4-x)$



Inverse Functions and Composition

ESSENTIALS

Composition and Inverses

If a function f is one-to-one, then f^{-1} is the unique function for which

$$\left.\begin{aligned}(f^{-1}\circ f)(x) &= f^{-1}(f(x)) = x \text{ and}\\ (f\circ f^{-1})(x) &= f(f^{-1}(x)) = x.\end{aligned}\right\} \text{cancellation equations}$$

$f^{-1}(f(x)) = x$

$f(f^{-1}(x)) = x$

$f(x) = \sqrt[3]{x+2},\ f^{-1}(x) = x^3 - 2$

① $f^{-1}(f(x)) = x$

$f^{-1}(f(x)) = f^{-1}\left(\sqrt[3]{x+2}\right)$

$= \left(\sqrt[3]{x+2}\right)^3 - 2$

$= x+2-2 = x$ ✓

② Check $f(f^{-1}(x)) = x$

$f(f^{-1}(x)) = f(x^3-2)$

$= \sqrt[3]{\underbrace{x^3-2}_{\text{input to } f}+2} = \sqrt[3]{x^3} = x$ ✓

Correct Inverse

$f(x) = x^3 + 5,\ f^{-1}(x) = \sqrt[3]{x-3}$

Check $f(f^{-1}(x)) = x$

$f(f^{-1}(x)) = f\left(\sqrt[3]{x-3}\right)$

$= \left(\sqrt[3]{x-3}\right)^3 + 5$

$= x-3+5$

$= x+2 \neq x$

$\Rightarrow$ Not the correct inverse

Quiz 11

① Let $f(x) = 2(x-3)^3$.

6 Pts Find a formula for $f^{-1}(x)$.

② Suppose you have to fence a rectangular field with perimeter of 40 ft. Find the dimensions which maximize the area.

7 Pts

③ The difference of two numbers is 10. Find the numbers if their Product is minimum.

7 Pts

$f(x) = ax^2 + bx + c \Rightarrow$ min/max occurs at $x = \frac{-b}{2a}$