1. Lines in Two-Dimensional Space

Problem 1: Describe and sketch all possible types of intersections of two lines in $\mathbb{R}^2$. Give equations of a pair of lines that illustrate each type of intersection.

Solution:

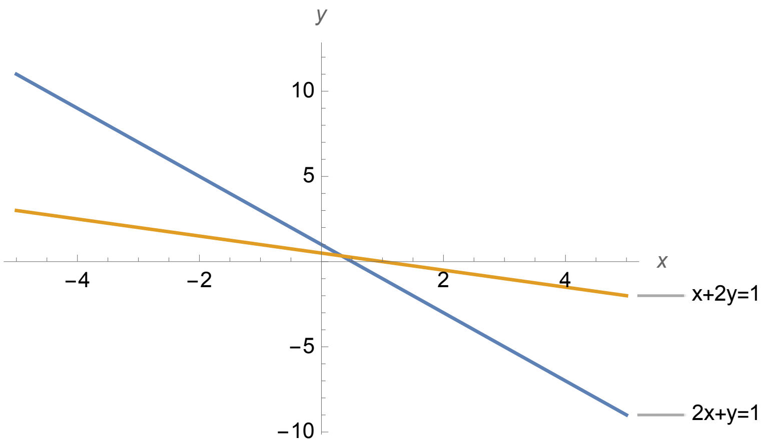


Figure 1: Unique Solution: A pair of line intersecting at a point

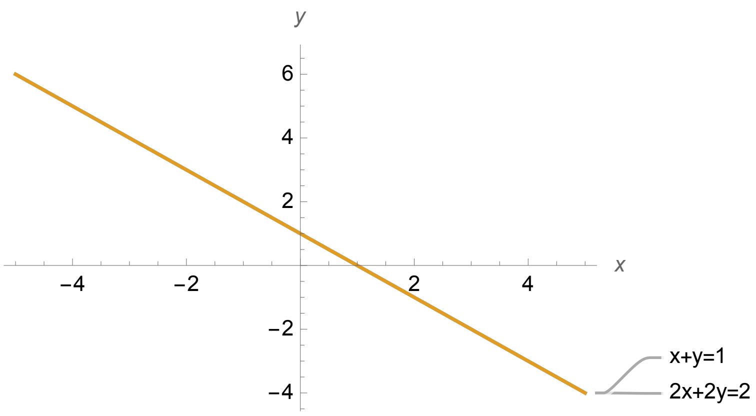


Figure 2: Infinite Solutions: A pair of coinciding lines

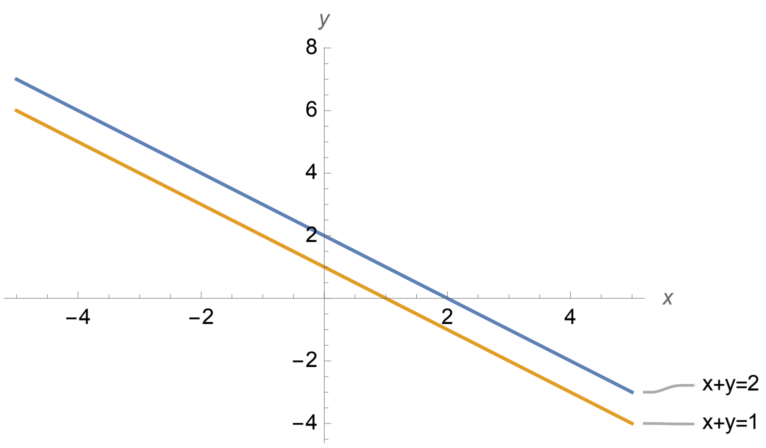
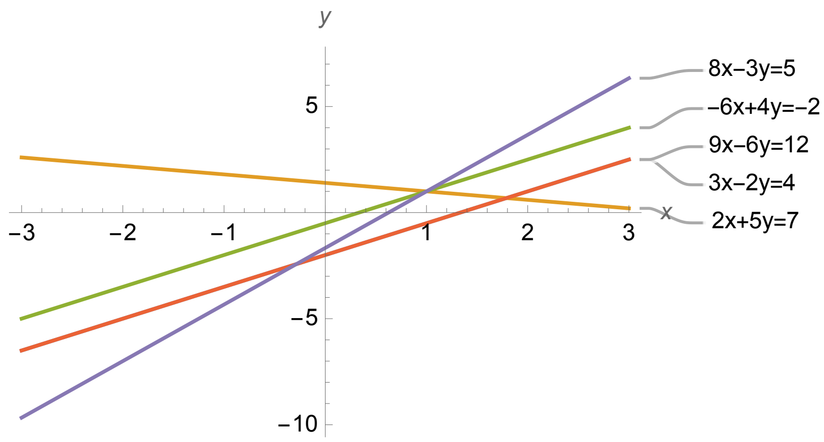


Figure 3: No Solution: A pair of parallel lines

Problem 2: On one page graph all of the the lines below. Then for each pair in the list eliminate a variable to determine if the pair define the same line, are parallel or intersect at one point. If a pair intersects at one point, find the point.

1. $3x - 2y = 4$
2. $2x + 5y = 7$
3. $-6x + 4y = -2$
4. $9x - 6y = 12$
5. $8x - 3y = 5$.

Solution:



Consider the pair $3x - 2y = 4$ and $2x + 5y = 7$:-
The lines intersect at the point $(34/19, 13/19)$.

Consider the pair $3x - 2y = 4$ and $-6x + 4y = -2$:-
The lines do not intersect at all and hence are parallel.

Consider the pair $3x - 2y = 4$ and $9x - 6y = 12$:-
The lines coincide with each other and hence both the equations have infinitely many solutions.

Consider the pair $3x - 2y = 4$ and $8x - 3y = 5$:-
The lines intersect at the point $(-2/7, -17/7)$.

Consider the pair $2x + 5y = 7$ and $-6x + 4y = -2$:-
The lines intersect at the point $(1, 1)$.

Consider the pair $2x + 5y = 7$ and $9x - 6y = 12$:-
The two given lines intersect at $(34/19, 13/19)$.

Consider the pair $2x + 5y = 7$ and $8x - 3y = 5$:-
The two given lines intersect at the point $(1, 1)$.

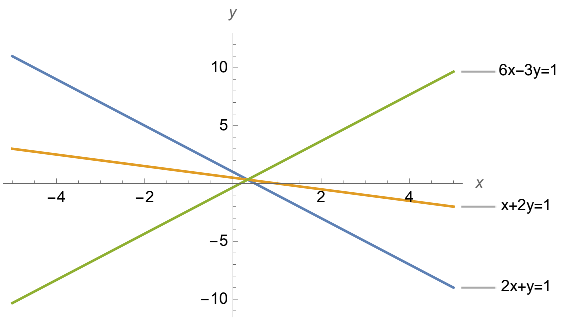
Consider the pair $-6x + 4y = -2$ and $9x - 6y = 12$:-
The lines do not intersect at all and hence are parallel.

Consider the pair $-6x + 4y = -2$ and $8x - 3y = 5$:-
The given lines intersect at the point $(1, 1)$.

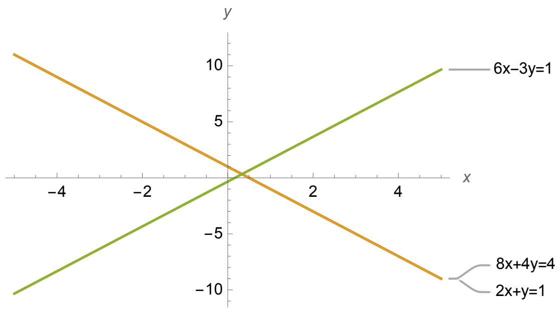
Consider the pair $9x - 6y = 12$ and $8x - 3y = 5$:-
The two given lines intersect at the point $(-2/7, -17/7)$.

Problem 3: Describe and sketch all possible types of intersections of three lines in $\mathbb{R}^2$. Give equations of a triple of lines that illustrate each type of intersection.

Solution: We have the following 7 cases:-

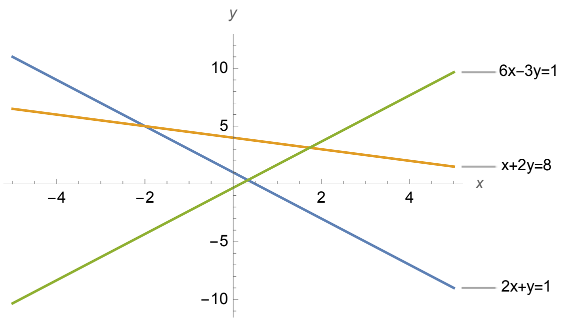


(a) Concurrent Lines

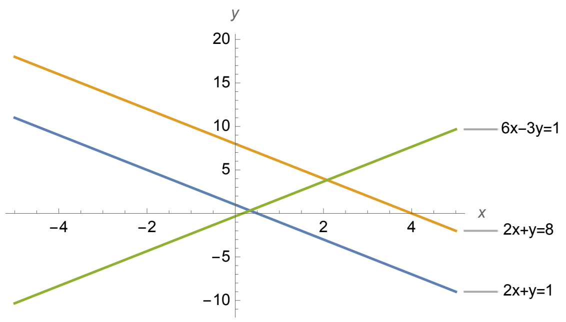


(b) Line intersecting two Coinciding Lines

Figure 4: Unique Solution

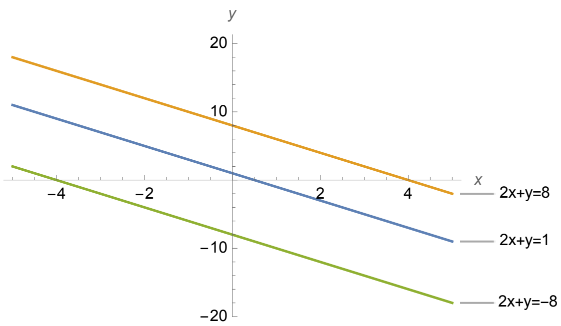


(a) Lines Forming a Triangle

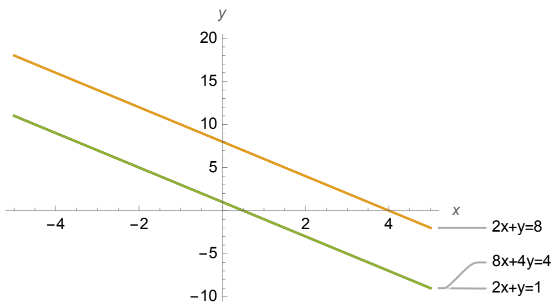


(b) Two Parallel Lines and a Transversal

Figure 5: No Solution



(a) Three Parallel Lines



(b) Two Coinciding and One Parallel

Figure 6: No Solution

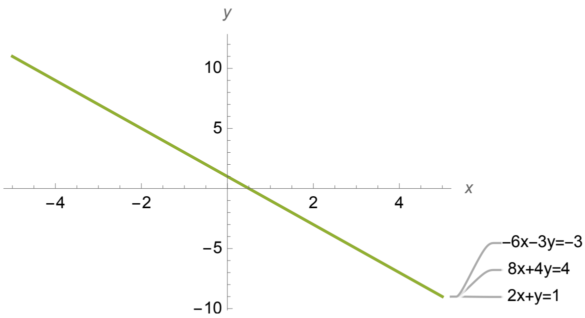


Figure 7: Infinitely Many Solutions: Three Coinciding Lines

Problem 4: Eliminate variables to determine if each triple of lines in the list above, intersect or do not intersect. If a triple intersects at one point find the point.

Solution:

$3x-2y=4, 2x+5y=7, -6x+4y=-2$	–	Do not intersect: Two parallel, one transversal
$3x-2y=4, 2x+5y=7, 9x-6y=12$	–	Line intersecting two coinciding lines at $(34/19, 13/19)$
$3x-2y=4, 2x+5y=7, 8x-3y=5$	–	Do not intersect: Form a triangle
$3x-2y=4, -6x+4y=-2, 9x-6y=12$	–	Do not intersect: Two coinciding, one parallel
$3x-2y=4, -6x+4y=-2, 8x-3y=5$	–	Do not intersect: Two parallel, one transversal
$3x-2y=4, 9x-6y=12, 8x-3y=5$	–	Line intersecting two coinciding lines at $(-2/7, -17/7)$
$2x+5y=7, -6x+4y=-2, 9x-6y=12$	–	Do not intersect: Two parallel, one transversal
$2x+5y=7, -6x+4y=-2, 8x-3y=5$	–	Concurrent Lines intersecting at $(1, 1, 1)$
$2x+5y=7, 9x-6y=12, 8x-3y=5$	–	Do not intersect: Form a triangle
$-6x+4y=-2, 9x-6y=12, 8x-3y=5$	–	Do not intersect: Two parallel, one transversal

2. Planes in Three-Dimensional Space

Problem 1: Describe and sketch all possible types of intersections of two planes in $\mathbb{R}^3$.

Solution:

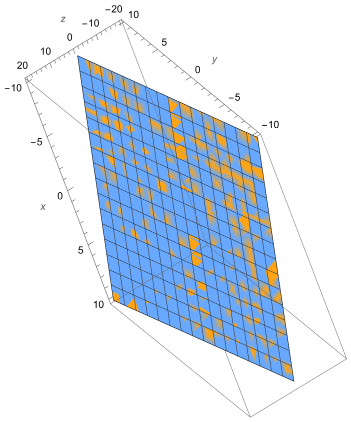


Figure 8: Two Coinciding Planes

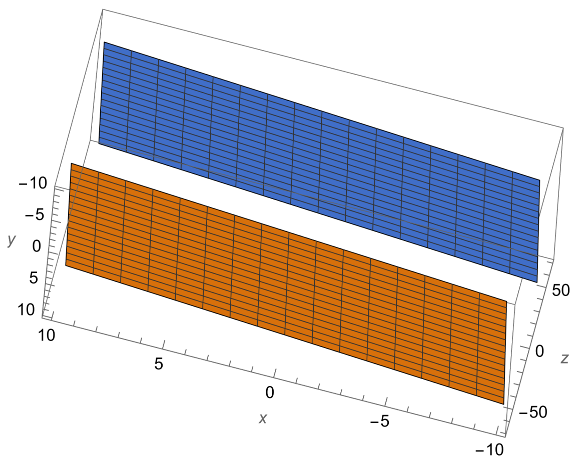


Figure 9: Two Parallel Planes

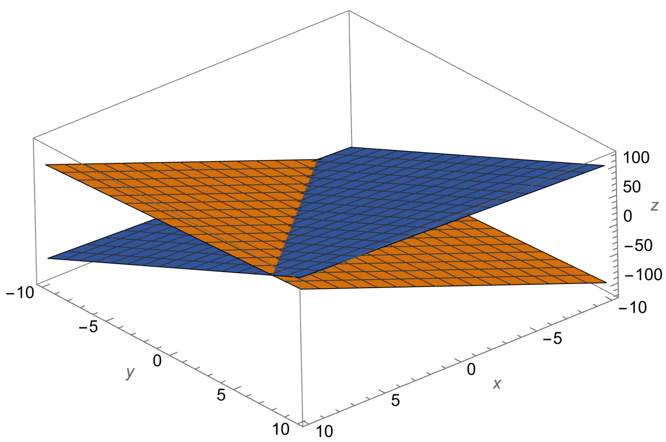


Figure 10: Two Planes Intersecting in a Line

Problem 2: Eliminate variables to determine if each pair of planes in the list below, define the same plane, intersect in a line or do not intersect. If the pair intersects in a line find parametric equations for the line of intersection.

1. $3x - 2y + z = 2$
2. $2x + y - z = 2$
3. $-x + 3y + 2z = 4$
4. $4x + 2y - 2z = 4$
5. $-6x + 4y - 2z = -2$

Solution: For all the pairs, eliminate x.

Consider the pair $3x - 2y + z = 2$ and $2x + y - z = 2$:-
The parametric equation of the line of intersection is $\boxed{x = (6+t)/7,\ y = (2+5t)/7,\ z = t}$.

Consider the pair $3x - 2y + z = 2$ and $-x + 3y + 2z = 4$:-
The parametric equation of the line of intersection is $\boxed{x = 2-t,\ y = 2-t,\ z = t}$.

Consider the pair $3x - 2y + z = 2$ and $4x + 2y - 2z = 4$:-
The parametric equation of the line of intersection is $\boxed{x = (6+t)/7,\ y = (2+5t)/7,\ z = t}$.

Consider the pair $3x - 2y + z = 2$ and $-6x + 4y - 2z = -2$:-
The given planes $\boxed{\text{do not intersect}}$ at all and hence are parallel.

Consider the pair $2x + y - z = 2$ and $-x + 3y + 2z = 4$:-
The parametric equation of the line of intersection is $\boxed{x = (2+5t)/7,\ y = (10-3t)/7,\ z = t}$.

Consider the pair $2x + y - z = 2$ and $4x + 2y - 2z = 4$:-
The given two planes $\boxed{\text{coincide}}$ with each other.

Consider the pair $2x + y - z = 2$ and $-6x + 4y - 2z = -2$:-
The parametric equation of the line of intersection is $\boxed{x = (5+t)/7,\ y = (4+5t)/7,\ z = t}$.

Consider the pair $-x + 3y + 2z = 4$ and $4x + 2y - 2z = 4$:-
The parametric equation of the line of intersection is $\boxed{x = (2+5t)/7,\ y = (10-3t)/7,\ z = t}$.

Consider the pair $-x + 3y + 2z = 4$ and $-6x + 4y - 2z = -2$:-
The parametric equation of the line of intersection is $\boxed{x = (11-7t)/7,\ y = (13-7t)/7,\ z = t}$.

Consider the pair $4x + 2y - 2z = 4$ and $-6x + 4y - 2z = -2$:-
The parametric equation of the line of intersection is $\boxed{x = (5+t)/7,\ y = (4+5t)/7,\ z = t}$.

Problem 3: Eliminate variables to determine if each triple of planes in problem 2, define the same plane, intersect in a line, intersect in one point or do not intersect. If the triple intersects in a line find parametric equations for the line of intersection. If the triple intersect at one point find the point.

Solution: Consider the triple $3x - 2y + z = 2$, $2x + y - z = 2$, $-x + 3y + 2z = 4$:-
The given triple of planes intersect at a unique point $\boxed{(1, 1, 1)}$.

Consider the triple $3x - 2y + z = 2$, $2x + y - z = 2$, $4x + 2y - 2z = 4$:-
The given triple intersects in a line whose parametric equation is $\boxed{x = (6+t)/7,\ y = (5+2t)/7,\ z = t}$.

Consider the triple $3x - 2y + z = 2$, $2x + y - z = 2$, $-6x + 4y - 2z = -2$:-
The given triple $\boxed{\text{do not intersect}}$.

Consider the triple $3x - 2y + z = 2$, $-x + 3y + 2z = 4$, $4x + 2y - 2z = 4$:-
The given planes intersect at a unique point $\boxed{(1, 1, 1)}$.

Consider the triple $3x - 2y + z = 2$, $-x + 3y + 2z = 4$, $-6x + 4y - 2z = -2$:-
The given planes $\boxed{\text{do not intersect}}$.

Consider the triple $3x - 2y + z = 2$, $4x + 2y - 2z = 4$, $-6x + 4y - 2z = -2$:-
The given planes $\boxed{\text{do not intersect}}$.

Consider the triple $2x + y - z = 2$, $-x + 3y + 2z = 4$, $4x + 2y - 2z = 4$:-
The given planes intersect in a line $\boxed{x = (2 + 5t)/7,\ y = (10 - 3t)/7,\ z = t}$.

Consider the triple $2x + y - z = 2$, $-x + 3y + 2z = 4$, $-6x + 4y - 2z = -2$:-
The given planes intersect at the point $\boxed{(23/28, 31/28, 3/4)}$.

Consider the triple $2x + y - z = 2$, $4x + 2y - 2z = 4$, $-6x + 4y - 2z = -2$:-
The given planes intersect in a line whose parametric equation is $\boxed{x = (5 + t)/7,\ y = (4 + 5t)/7,\ z = t}$.

Consider the triple $-x + 3y + 2z = 4$, $4x + 2y - 2z = 4$, $-6x + 4y - 2z = -2$:-
The given planes intersect at the point $\boxed{(23/28, 31/28, 3/4)}$.

Problem 4: Use the parametric equations for the line of intersection of the planes (1) and (2) in problem 2 to find the intersection of this line with the planes defined by (3), (4) and (5). Compare this with the answers in problem 3 you obtained by eliminating variables.

Solution: The parametric equation for the line of intersection of planes (1) and (2) is
$x = (6 + t)/7$, $y = (2 + 5t)/7$, $z = t$.

Intersection with plane (3) $-x + 3y + 2z = 4$:-
$-(6+t)/7+3(2+5t)/7+2t = 4 \Rightarrow 28t = 28 \Rightarrow t = 1 \Rightarrow x = (6+1)/7 = 1$, $y = (2+5)/7 = 1$, $z = 1$. Thus, planes (1), (2), (3) intersect at the point $(1, 1, 1)$.

Intersection with plane (4) $4x + 2y - 2z = 4$:-
$4(6 + t)/7 + 2(2 + 5t)/7 - 2t = 4 \Rightarrow 0 = 0$. Thus, planes (1), (2), (4) intersect in the line $x = (6 + t)/7$, $y = (2 + 5t)/7$, $z = t$.

Intersection with plane (5) $-6x + 4y - 2z = -2$:-
$-6(6 + t)/7 + 4(2 + 5t)/7 - 2t = -2 \Rightarrow 0 = 2$. Thus, the planes (1), (2), (5) do not intersect.

3. Matrices and Elementary Row Operations

Problem 1: For each pair of equations in problem 2, section 1 (Lines in two-dimensional space), write down the associated augmented matrix, the coefficient matrix and the constant vector.

Solution: For the pair $3x - 2y = 4$, $2x + 5y = 7$ the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[\begin{array}{cc|c} 3 & -2 & 4 \\ 2 & 5 & 7 \end{array}\right] \quad , \quad \begin{bmatrix} 3 & -2 \\ 2 & 5 \end{bmatrix} \quad , \quad \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

For the pair $3x - 2y = 4$ and $-6x + 4y = -2$ the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[\begin{array}{cc|c} 3 & -2 & 4 \\ -6 & 4 & -2 \end{array}\right] \quad , \quad \begin{bmatrix} 3 & -2 \\ -6 & 4 \end{bmatrix} \quad , \quad \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

For the pair $3x - 2y = 4$ and $9x - 6y = 12$ the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[\begin{array}{cc|c} 3 & -2 & 4 \\ 9 & -6 & 12 \end{array}\right] \quad , \quad \begin{bmatrix} 3 & -2 \\ 9 & -6 \end{bmatrix} \quad , \quad \begin{bmatrix} 4 \\ 12 \end{bmatrix}$$

For the pair $3x - 2y = 4$ and $8x - 3y = 5$ the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[\begin{array}{cc|c} 3 & -2 & 4 \\ 8 & -3 & 5 \end{array}\right] \quad , \quad \begin{bmatrix} 3 & -2 \\ 8 & -3 \end{bmatrix} \quad , \quad \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

For the pair $2x + 5y = 7$ and $-6x + 4y = -2$ the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[\begin{array}{cc|c} 2 & 5 & 7 \\ -6 & 4 & -2 \end{array}\right] \quad , \quad \begin{bmatrix} 2 & 5 \\ -6 & 4 \end{bmatrix} \quad , \quad \begin{bmatrix} 7 \\ -2 \end{bmatrix}$$

For the pair $2x + 5y = 7$ and $9x - 6y = 12$ the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[\begin{array}{cc|c} 2 & 5 & 7 \\ 9 & -6 & 12 \end{array}\right] \quad , \quad \begin{bmatrix} 2 & 5 \\ 9 & -6 \end{bmatrix} \quad , \quad \begin{bmatrix} 7 \\ 12 \end{bmatrix}$$

For the pair $2x + 5y = 7$ and $8x - 3y = 5$ the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[\begin{array}{cc|c} 2 & 5 & 7 \\ 8 & -3 & 5 \end{array}\right] \quad , \quad \begin{bmatrix} 2 & 5 \\ 8 & -3 \end{bmatrix} \quad , \quad \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

For the pair $-6x+4y=-2$ and $9x-6y=12$ the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[\begin{array}{cc|c} -6 & 4 & -2 \\ 9 & -6 & 12 \end{array}\right] \quad , \quad \begin{bmatrix} -6 & 4 \\ 9 & -6 \end{bmatrix} \quad , \quad \begin{bmatrix} -2 \\ 12 \end{bmatrix}$$

For the pair $-6x+4y=-2$ and $8x-3y=5$ the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[\begin{array}{cc|c} -6 & 4 & -2 \\ 8 & -3 & 5 \end{array}\right] \quad , \quad \begin{bmatrix} -6 & 4 \\ 8 & -3 \end{bmatrix} \quad , \quad \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

For the pair $9x-6y=12$ and $8x-3y=5$ the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[\begin{array}{cc|c} 9 & -6 & 12 \\ 8 & -3 & 5 \end{array}\right] \quad , \quad \begin{bmatrix} 9 & -6 \\ 8 & -3 \end{bmatrix} \quad , \quad \begin{bmatrix} 12 \\ 5 \end{bmatrix}$$

Problem 2: For each triple of equations in problem 2, section 2 (Planes in three-dimensional space), write down the associated augmented matrix, the coefficient matrix and the constant vector.

Solution: For the triple $3x-2y+z=2$, $2x+y-z=2$, $-x+3y+2z=4$ the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & 2 \\ 2 & 1 & -1 & 2 \\ -1 & 3 & 2 & 4 \end{array}\right] \quad , \quad \begin{bmatrix} 3 & -2 & 1 \\ 2 & 1 & -1 \\ -1 & 3 & 2 \end{bmatrix} \quad , \quad \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$

For the triple $3x-2y+z=2$, $2x+y-z=2$, $4x+2y-2z=4$ the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & 2 \\ 2 & 1 & -1 & 2 \\ 4 & 2 & -2 & 4 \end{array}\right] \quad , \quad \begin{bmatrix} 3 & -2 & 1 \\ 2 & 1 & -1 \\ 4 & 2 & -2 \end{bmatrix} \quad , \quad \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$

For the triple $3x-2y+z=2$, $2x+y-z=2$, $-6x+4y-2z=-2$ the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & 2 \\ 2 & 1 & -1 & 2 \\ -6 & 4 & -2 & -2 \end{array}\right] \quad , \quad \begin{bmatrix} 3 & -2 & 1 \\ 2 & 1 & -1 \\ -6 & 4 & -2 \end{bmatrix} \quad , \quad \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}$$

For the triple $3x-2y+z=2$, $-x+3y+2z=4$, $4x+2y-2z=4$ the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & 2 \\ -1 & 3 & 2 & 4 \\ 4 & 2 & -2 & 4 \end{array}\right] \quad , \quad \begin{bmatrix} 3 & -2 & 1 \\ -1 & 3 & 2 \\ 4 & 2 & -2 \end{bmatrix} \quad , \quad \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$$

For the triple $3x - 2y + z = 2$, $-x + 3y + 2z = 4$, $-6x + 4y - 2z = -2$ the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & 2 \\ -1 & 3 & 2 & 4 \\ -6 & 4 & -2 & -2 \end{array}\right] \quad , \quad \begin{bmatrix} 3 & -2 & 1 \\ -1 & 3 & 2 \\ -6 & 4 & -2 \end{bmatrix} \quad , \quad \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$$

For the triple $3x - 2y + z = 2$, $4x + 2y - 2z = 4$, $-6x + 4y - 2z = -2$ the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & 2 \\ 4 & 2 & -2 & 4 \\ -6 & 4 & -2 & -2 \end{array}\right] \quad , \quad \begin{bmatrix} 3 & -2 & 1 \\ 4 & 2 & -2 \\ -6 & 4 & -2 \end{bmatrix} \quad , \quad \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$$

For the triple $2x + y - z = 2$, $-x + 3y + 2z = 4$, $4x + 2y - 2z = 4$ the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ -1 & 3 & 2 & 4 \\ 4 & 2 & -2 & 4 \end{array}\right] \quad , \quad \begin{bmatrix} 2 & 1 & -1 \\ -1 & 3 & 2 \\ 4 & 2 & 2 \end{bmatrix} \quad , \quad \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$$

For the triple $2x + y - z = 2$, $-x + 3y + 2z = 4$, $-6x + 4y - 2z = -2$ the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ -1 & 3 & 2 & 4 \\ -6 & 4 & -2 & -2 \end{array}\right] \quad , \quad \begin{bmatrix} 2 & 1 & -1 \\ -1 & 3 & 2 \\ -6 & 4 & -2 \end{bmatrix} \quad , \quad \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$$

For the triple $2x + y - z = 2$, $4x + 2y - 2z = 4$, $-6x + 4y - 2z = -2$ the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 4 & 2 & -2 & 4 \\ -6 & 4 & -2 & -2 \end{array}\right] \quad , \quad \begin{bmatrix} 2 & 1 & -1 \\ 4 & 2 & -2 \\ -6 & 4 & -2 \end{bmatrix} \quad , \quad \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$$

For the triple $-x + 3y + 2z = 4$, $4x + 2y - 2z = 4$, $-6x + 4y - 2z = -2$ the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[\begin{array}{ccc|c} -1 & 3 & 2 & 4 \\ 4 & 2 & -2 & 4 \\ -6 & 4 & -2 & -2 \end{array}\right] \quad , \quad \begin{bmatrix} -1 & 3 & 2 \\ 4 & 2 & -2 \\ -6 & 4 & -2 \end{bmatrix} \quad , \quad \begin{bmatrix} 4 \\ 4 \\ -2 \end{bmatrix}$$

Problem 3: Determine which of the following matrices are in reduced row-echelon form.

Solution:

1.
$$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \to \text{No} \quad \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \to \text{No} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to \text{Yes} \quad \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \to \text{No}$$

$$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} \to \text{Yes} \quad \begin{bmatrix}1 & 0\\0 & 2\end{bmatrix} \to \text{No} \quad \begin{bmatrix}1 & 2\\0 & 0\end{bmatrix} \to \text{Yes} \quad \begin{bmatrix}2 & 0\\0 & 1\end{bmatrix} \to \text{No}$$

2.

$$\begin{bmatrix}1 & 0 & 2\\0 & 1 & 1\end{bmatrix} \to \text{Yes} \quad \begin{bmatrix}1 & 2 & 0\\0 & 1 & 1\end{bmatrix} \to \text{No} \quad \begin{bmatrix}1 & 2 & 0\\0 & 0 & 1\end{bmatrix} \to \text{Yes}$$

$$\begin{bmatrix}0 & 1 & 1\\1 & 0 & 2\end{bmatrix} \to \text{No} \quad \begin{bmatrix}1 & 2 & 3\\0 & 0 & 0\end{bmatrix} \to \text{Yes} \quad \begin{bmatrix}1 & 2 & 0\\0 & 0 & -1\end{bmatrix} \to \text{No}$$

3.

$$\begin{bmatrix}1 & 1 & 1\\0 & 1 & 2\\0 & 0 & 0\end{bmatrix} \to \text{No} \quad \begin{bmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \to \text{No} \quad \begin{bmatrix}1 & 1 & 2\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix} \to \text{No}$$

$$\begin{bmatrix}1 & 0 & 1\\0 & 1 & 2\\0 & 0 & 0\end{bmatrix} \to \text{Yes} \quad \begin{bmatrix}0 & 0 & 0\\1 & 1 & 1\\0 & 1 & 2\end{bmatrix} \to \text{No} \quad \begin{bmatrix}1 & 2 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix} \to \text{Yes}$$

4.

$$\begin{bmatrix}1 & 1 & 1 & 0\\0 & 1 & 2 & 0\\0 & 0 & 0 & 1\end{bmatrix} \to \text{No} \quad \begin{bmatrix}1 & 2 & 1 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{bmatrix} \to \text{Yes} \quad \begin{bmatrix}1 & 1 & 1 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 1\end{bmatrix} \to \text{No}$$

$$\begin{bmatrix}1 & 1 & 0 & 2\\0 & 0 & 1 & 3\\0 & 0 & 0 & 0\end{bmatrix} \to \text{Yes} \quad \begin{bmatrix}1 & 0 & 0 & 2\\0 & 0 & 1 & 3\\0 & 1 & 0 & 0\end{bmatrix} \to \text{No}$$

Problem 4: There are 15 leading 1 patterns for 3×4 matrices in reduced row-echelon form. Write down all of them.

Solution:

$$\begin{bmatrix}1 & 0 & 0 & *\\0 & 1 & 0 & *\\0 & 0 & 1 & *\end{bmatrix} \quad \begin{bmatrix}1 & 0 & * & 0\\0 & 1 & * & 0\\0 & 0 & 0 & 1\end{bmatrix} \quad \begin{bmatrix}1 & * & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix} \quad \begin{bmatrix}0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix} \quad \begin{bmatrix}1 & 0 & * & *\\0 & 1 & * & *\\0 & 0 & 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & * & 0 & *\\0 & 0 & 1 & *\\0 & 0 & 0 & 0\end{bmatrix} \quad \begin{bmatrix}0 & 1 & 0 & *\\0 & 0 & 1 & *\\0 & 0 & 0 & 0\end{bmatrix} \quad \begin{bmatrix}1 & * & * & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{bmatrix} \quad \begin{bmatrix}0 & 1 & * & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{bmatrix} \quad \begin{bmatrix}0 & 0 & 1 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & * & * & *\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix} \quad \begin{bmatrix}0 & 1 & * & *\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix} \quad \begin{bmatrix}0 & 0 & 1 & *\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix} \quad \begin{bmatrix}0 & 0 & 0 & 1\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix} \quad \begin{bmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}$$

4. Gaussian Elimination

Problem 1: Use Gaussian elimination to put the following matrices in reduced row-echelon form.

1. $\left[\begin{array}{cccc|c} 3 & 3 & -4 & -2 & 1 \\ 2 & 2 & -3 & 1 & 3 \\ 1 & 1 & -2 & 4 & 5 \end{array}\right]$

Solution:

$$\left[\begin{array}{cccc|c} 3 & 3 & -4 & -2 & 1 \\ 2 & 2 & -3 & 1 & 3 \\ 1 & 1 & -2 & 4 & 5 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{cccc|c} \boxed{1} & 1 & -2 & 4 & 5 \\ 2 & 2 & -3 & 1 & 3 \\ 3 & 3 & -4 & -2 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{cccc|c} \boxed{1} & 1 & -2 & 4 & 5 \\ 0 & 0 & 1 & -7 & -7 \\ 3 & 3 & -4 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 3R_1} \left[\begin{array}{cccc|c} \boxed{1} & 1 & -2 & 4 & 5 \\ 0 & 0 & \boxed{1} & -7 & -7 \\ 0 & 0 & 2 & -14 & -14 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_2]{R_1 \to R_1 + 2R_2} \left[\begin{array}{cccc|c} \boxed{1} & 1 & 0 & -10 & -9 \\ 0 & 0 & \boxed{1} & -7 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

2. $\left[\begin{array}{cccc|c} 2 & 3 & 3 & -1 & 3 \\ 1 & 1 & -2 & 3 & 4 \\ 5 & 7 & 4 & 1 & 5 \end{array}\right]$

Solution:

$$\left[\begin{array}{cccc|c} 2 & 3 & 3 & -1 & 3 \\ 1 & 1 & -2 & 3 & 4 \\ 5 & 7 & 4 & 1 & 5 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cccc|c} \boxed{1} & 1 & -2 & 3 & 4 \\ 2 & 3 & 3 & -1 & 3 \\ 5 & 7 & 4 & 1 & 5 \end{array}\right] \xrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{cccc|c} \boxed{1} & 1 & -2 & 3 & 4 \\ 0 & \boxed{1} & 7 & -7 & -5 \\ 0 & 2 & 14 & -14 & -15 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - 2R_2]{R_1 \to R_1 - R_2} \left[\begin{array}{cccc|c} \boxed{1} & 0 & -9 & 10 & 9 \\ 0 & \boxed{1} & 7 & -7 & -5 \\ 0 & 0 & 0 & 0 & -5 \end{array}\right] \xrightarrow{R_3 \to (-1/5)R_3} \left[\begin{array}{cccc|c} \boxed{1} & 0 & -9 & 10 & 9 \\ 0 & \boxed{1} & 7 & -7 & -5 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 + 5R_3]{R_1 \to R_1 - 9R_3} \left[\begin{array}{cccc|c} \boxed{1} & 0 & -9 & 10 & 0 \\ 0 & \boxed{1} & 7 & -7 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{array}\right]$$

3. $\left[\begin{array}{cc|c} 2 & -4 & 6 \\ -2 & 3 & -3 \\ 3 & 7 & 5 \end{array}\right]$

Solution:

$$\left[\begin{array}{cc|c} 2 & -4 & 6 \\ -2 & 3 & -3 \\ 3 & 7 & 5 \end{array}\right] \xrightarrow{R_1 \to (1/2)R_1} \left[\begin{array}{cc|c} \boxed{1} & -2 & 3 \\ -2 & 3 & -3 \\ 3 & 7 & 5 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 + 2R_1} \left[\begin{array}{cc|c} \boxed{1} & -2 & 3 \\ 0 & -1 & 3 \\ 0 & 13 & -4 \end{array}\right] \xrightarrow{R_2 \to -R_2} \left[\begin{array}{cc|c} \boxed{1} & -2 & 3 \\ 0 & \boxed{1} & -3 \\ 0 & 13 & -4 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - 13R_2]{R_1 \to R_1 + 2R_2} \left[\begin{array}{cc|c} \boxed{1} & 0 & -3 \\ 0 & \boxed{1} & -3 \\ 0 & 0 & 35 \end{array}\right] \xrightarrow{R_3 \to (1/35)R_3} \left[\begin{array}{cc|c} \boxed{1} & 0 & -3 \\ 0 & \boxed{1} & -3 \\ 0 & 0 & \boxed{1} \end{array}\right] \xrightarrow[R_2 \to R_2 + 3R_3]{R_1 \to R_1 + 3R_3} \left[\begin{array}{cc|c} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{array}\right]$$

4. $\left[\begin{array}{ccc|c} 2 & -4 & 6 & 8 \\ 3 & 7 & 5 & 3 \\ -1 & -1 & 17 & 19 \end{array}\right]$

Solution:

$$\left[\begin{array}{ccc|c} 2 & -4 & 6 & 8 \\ 3 & 7 & 5 & 3 \\ -1 & -1 & 17 & 19 \end{array}\right] \xrightarrow{R_1 \to (1/2)R_1} \left[\begin{array}{ccc|c} \boxed{1} & -2 & 3 & 4 \\ 3 & 7 & 5 & 3 \\ -1 & -1 & 17 & 19 \end{array}\right] \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 3R_1} \left[\begin{array}{ccc|c} \boxed{1} & -2 & 3 & 4 \\ 0 & 13 & -4 & -9 \\ 0 & -3 & 20 & 23 \end{array}\right]$$

$$\xrightarrow{R_2 \to (1/13)R_2} \left[\begin{array}{ccc|c} \boxed{1} & -2 & 3 & 4 \\ 0 & \boxed{1} & -4/13 & -9/13 \\ 0 & -3 & 20 & 23 \end{array}\right] \xrightarrow[R_3 \to R_3 + 3R_2]{R_1 \to R_1 + 2R_2} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 31/13 & 34/13 \\ 0 & \boxed{1} & -4/13 & -9/13 \\ 0 & 0 & 248/13 & 272/13 \end{array}\right]$$

$$\xrightarrow{R_3 \to (13/248)R_3} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 31/13 & 34/13 \\ 0 & \boxed{1} & -4/13 & -9/13 \\ 0 & 0 & \boxed{1} & 34/31 \end{array}\right] \xrightarrow[R_2 \to R_2 + (4/13)R_3]{R_1 \to R_1 - (31/13)R_3} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & -11/13 \\ 0 & 0 & \boxed{1} & 34/31 \end{array}\right]$$

5. $\left[\begin{array}{ccccc|c} -3 & -6 & 9 & 6 & -12 & -3 \\ 2 & 5 & -8 & -1 & 6 & 4 \\ 1 & 2 & -3 & -2 & 4 & 1 \end{array}\right]$

Solution:

$$\left[\begin{array}{ccccc|c} -3 & -6 & 9 & 6 & -12 & -3 \\ 2 & 5 & -8 & -1 & 6 & 4 \\ 1 & 2 & -3 & -2 & 4 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccccc|c} \boxed{1} & 2 & -3 & -2 & 4 & 1 \\ 2 & 5 & -8 & -1 & 6 & 4 \\ -3 & -6 & 9 & 6 & -12 & -3 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 - 2R_1]{R_3 \to R_3 + 3R_1} \left[\begin{array}{ccccc|c} \boxed{1} & 2 & -3 & -2 & 4 & 1 \\ 0 & \boxed{1} & -2 & 3 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{ccccc|c} \boxed{1} & 0 & 1 & -8 & 8 & -3 \\ 0 & \boxed{1} & -2 & 3 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

5. Reduced Row Echelon Matrices and Solution Sets

Problem 1: Use elementary row operations to put each matrix from problem 1, section 3 (Matrices and elementary row operations), in reduced row-echelon form. Use this to write down the solution set and compare the answer to the ones you obtained in problem 2, section 1 (Lines in two-dimensional space).

Solution: For the pair $3x - 2y = 4$, $2x + 5y = 7$ we have

$$\left[\begin{array}{cc|c} 3 & -2 & 4 \\ 2 & 5 & 7 \end{array}\right] \xrightarrow{R_1 \to (1/3)R_1} \left[\begin{array}{cc|c} \boxed{1} & -2/3 & 4/3 \\ 2 & 5 & 7 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{cc|c} \boxed{1} & -2/3 & 4/3 \\ 0 & 19/3 & 13/3 \end{array}\right]$$

$$\xrightarrow{R_2 \to (3/19)R_2} \left[\begin{array}{cc|c} \boxed{1} & -2/3 & 4/3 \\ 0 & \boxed{1} & 13/19 \end{array}\right] \xrightarrow{R_1 \to R_1 + (2/3)R_2} \begin{array}{c} \left[\begin{array}{cc|c} \boxed{1} & 0 & 34/19 \\ 0 & \boxed{1} & 39/19 \end{array}\right] \\ \begin{array}{cc} \uparrow & \uparrow \\ x & y \end{array} \end{array}$$

Both x and y are basic variables. The solution set is thus the single point $\left(\dfrac{34}{19}, \dfrac{13}{19}\right)$

Problem 2: Use elementary row operations to put each matrix from problem 2, section 3 (Matrices and elementary row operations), in reduced row-echelon form. Use this to write down the solution set and compare the answer to the ones you obtained in problem 2, section 2 (Planes in two-dimensional space).

Solution: For the triple $3x - 2y + z = 2$, $2x + y - z = 2$, $-x + 3y + 2z = 4$ we have

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & 2 \\ 2 & 1 & -1 & 2 \\ -1 & 3 & 2 & 4 \end{array}\right] \xrightarrow{R_1 \to (1/3)R_1} \left[\begin{array}{ccc|c} \boxed{1} & -2/3 & 1/3 & 2/3 \\ 2 & 1 & -1 & 2 \\ -1 & 3 & 2 & 4 \end{array}\right] \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} \boxed{1} & -2/3 & 1/3 & 2/3 \\ 0 & 7/3 & -5/3 & 2/3 \\ 0 & 7/3 & 7/3 & 14/3 \end{array}\right]$$

$$\xrightarrow{R_2 \to (3/7)R_2} \left[\begin{array}{ccc|c} \boxed{1} & -2/3 & 1/3 & 2/3 \\ 0 & \boxed{1} & -5/7 & 2/7 \\ 0 & 7/3 & 7/3 & 14/3 \end{array}\right] \xrightarrow[R_3 \to R_3 - (7/3)R_2]{R_1 \to R_1 + (2/3)R_2} \left[\begin{array}{ccc|c} \boxed{1} & 0 & -1/7 & 6/7 \\ 0 & \boxed{1} & -5/7 & 2/7 \\ 0 & 0 & 4 & 4 \end{array}\right]$$

$$\xrightarrow{R_3 \to (1/4)R_3} \left[\begin{array}{ccc|c} \boxed{1} & 0 & -1/7 & 6/7 \\ 0 & \boxed{1} & -5/7 & 2/7 \\ 0 & 0 & \boxed{1} & 1 \end{array}\right] \xrightarrow[R_2 \to R_2 + (5/7)R_3]{R_1 \to R_1 + (1/7)R_3} \begin{array}{c} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{1} & 1 \end{array}\right] \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ x & y & z \end{array} \end{array}$$

All the variables x, y and z are basic. The solution set is thus the single point $(1, 1, 1)$.

Problem 3: For each reduced row-echelon matrix obtained in problem 1, section 4 (Gaussian elimination), write down the solution set for the system of linear equations associated to the original matrix.

Solution: (1) The reduced row-echelon form obtained was
$$\begin{array}{l} \left[\begin{array}{cccc|c} \boxed{1} & 1 & 0 & -10 & -9 \\ 0 & 0 & \boxed{1} & -7 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ x_1 & x_2 & x_3 & x_4 \end{array} \end{array}$$

The variables x_1 and x_3 are basic while the variables x_2 and x_4 are free.

So we let $x_2 = s$, $x_4 = t$ where $s, t \in \mathbb{R}$.

The second row gives $x_3 - 7x_4 = -7 \Rightarrow x_3 - 7t = -7 \Rightarrow x_3 = -7 + 7t$.

The first row gives $x_1 + x_2 - 10x_4 = -9 \Rightarrow x_1 + s - 10t = -9 \Rightarrow x_1 = -9 - s + 10t$.

Thus the solution set is given by

$$\boxed{x_1 = -9 - s + 10t \quad , \quad x_2 = s \quad , \quad x_3 = -7 + 7t \quad , \quad x_4 = t \quad , \quad s\,,\, t \in \mathbb{R}}$$

which is a plane.

(2) The reduced row-echelon form obtained was
$$\begin{array}{l} \left[\begin{array}{cccc|c} \boxed{1} & 0 & -9 & 10 & 0 \\ 0 & \boxed{1} & 7 & -7 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{array}\right] \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ x_1 & x_2 & x_3 & x_4 \end{array} \end{array}$$

Since we have a pivot position in the column corresponding to the constant vector, the solution set is empty. Thus, $\boxed{\text{no solutions}}$ exist for the given system.

(3) The reduced row-echelon form obtained was
$$\begin{array}{l} \left[\begin{array}{cc|c} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{array}\right] \\ \begin{array}{cc} \uparrow & \uparrow \\ x_1 & x_2 \end{array} \end{array}$$

Since we have a pivot position in the column corresponding to the constant vector, the solution set is empty. Thus, $\boxed{\text{no solutions}}$ exist for the given system.

(4) The reduced row-echelon form obtained was
$$\begin{array}{l} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & -11/13 \\ 0 & 0 & \boxed{1} & 34/31 \end{array}\right] \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ x_1 & x_2 & x_3 \end{array} \end{array}$$

All the variables x_1, x_2, x_3 are basic. The solution set is thus the single point $\left(0\,,\, -\dfrac{11}{13}, \dfrac{34}{31}\right)$.

(5) The reduced row-echelon form obtained was

$$\begin{array}{c} \left[\begin{array}{ccccc|c} \boxed{1} & 0 & 1 & -8 & 8 & -3 \\ 0 & \boxed{1} & -2 & 3 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \end{array}$$

The variables x_1 and x_2 are basic while the variables x_3, x_4 and x_5 are free.

So we let $x_3 = s$, $x_4 = t$, $x_5 = u$ where s, t, $u \in \mathbb{R}$.

The second row gives $x_2 - 2x_3 + 3x_4 - 2x_5 = 2 \Rightarrow x_2 - 2s + 3t - 2u = 2 \Rightarrow x_3 = 2 + 2s - 3t + 2u$.

The first row gives $x_1 + x_3 - 8x_4 + 8x_5 = -3 \Rightarrow x_1 + s - 8t + 8u = -3 \Rightarrow x_1 = -3 - s + 8t - 8u$.

Thus the solution set is given by

$$\boxed{x_1 = -3 - s + 8t - 8u \ , \ \ x_2 = 2 + 2s - 3t + 2u \ , \ \ x_3 = s \ , \ \ x_4 = t \ , \ \ x_5 = u \ , \ \ s, t, u \in \mathbb{R}}$$

which is a 3-dimensional linear space.

Problem 4: Each of the following matrices is in reduced row-echelon form and represents a system of linear equations. Write down the solution set for the system of equations associated to each matrix.

1. $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

   Solution: Since there is pivot in the column vector of constant terms, there is $\boxed{\text{no solution}}$.

2. $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

   Solution: The variable x_2 is free. So we let $x_2 = t$. The second row gives $x_3 = -3$. The first row gives $x_1 + 2x_2 = 0 \Rightarrow x_1 = -2t$. Thus, the solution set is $\boxed{\{(-2t, t, 0) : t \in \mathbb{R}\}}$

3. $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right]$

   Solution: The variable x_3 is free. So we let $x_3 = t$. The second row gives $x_2 + 4x_3 = -5$ $\Rightarrow x_2 = -5 - 4t$. The first row gives $x_1 - 2x_3 = 3 \Rightarrow x_1 = 3 + 2t$.
   Thus, the solution set is $\boxed{\{(3 + 2t, -5 - 4t, t) : t \in \mathbb{R}\}}$

4. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \end{array}\right]$

   Solution: All variables are basic. The solution set is the point $\boxed{(2, -3, 0)}$

5. $\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Solution: The variables x_2 and x_3 are free. So we let $x_2 = s$, $x_3 = t$.
The first row gives $x_1 + 2x_2 - 3x_3 = 4 \Rightarrow x_1 = 4 - 2s + 3t$.
Thus, the solution set is $\boxed{\{(4 - 2s + 3t, s, t) : s, t \in \mathbb{R}\}}$

6. $\left[\begin{array}{ccc|c} 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Solution: The variable x_1 is free. So we let $x_1 = t$. The second row gives $x_3 = 4$ and the first row gives $x_2 = -3$. Thus, the solution set is $\boxed{\{(t, -3, 4) : t \in \mathbb{R}\}}$

7. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$

Solution: All variables are basic. Thus, the solution set is the single point $\boxed{(2, 7, -3, 1)}$

8. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 3 & 0 & 7 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Solution: The variable x_3 is free. So we let $x_3 = t$. The third row gives $x_4 = -3$.
The second row gives $x_2 + 3x_3 = 7 \Rightarrow x_2 = 7 - 3t$. The first row gives $x_1 = 2$.
Thus, the solution set is $\boxed{\{(2, 7 - 3t, t, -3) : t \in \mathbb{R}\}}$

9. $\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Solution: The variable x_2 is free. So we let $x_2 = t$. The third row gives $x_4 = -3$.
The second row gives $x_3 = 7$. The first row gives $x_1 + 2x_2 = 2 \Rightarrow x_1 = 2 - 2t$.
Thus, the solution set is $\boxed{\{(2 - 2t, t, 7, -3) : t \in \mathbb{R}\}}$

10. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Solution: The variables x_3 and x_4 are free. So we let $x_3 = s$ and $x_4 = t$.
The second row gives $x_2 = 7$ and the first row gives $x_1 = 2$.
Thus, the solution set is $\boxed{(2, 7, s, t) : s, t \in \mathbb{R}\}}$

Problem 5: Suppose we start with a system of three linear equations in three variables, write down the associated augmented matrix, put it in reduced row-echelon form and find the solution set. For each solution set below, find the reduced row-echelon matrix it came from.

1. The solution set is the point $(-4, 3, 1)$.

Solution: $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$

2. The solution set is the line $x = 1 - 4t, y = 3 + 3t, z = 2 - t$.

   Solution: Let $z = 2 - t = s$. Then $t = 2 - s \Rightarrow y = 3 + 3t = 3 + 3(2 - s) = 9 - 3s$ $\Rightarrow x = 1 - 4t = 1 - 4(2 - s) = -7 + 4s$. Thus the given line can also be represented by the parametric equation $x = -7 + 4s$, $y = 9 - 3s$ and $z = s$. So the variable z is free and the corresponding reduced row-echelon matrix would be

$$\left[\begin{array}{ccc|c} 1 & 0 & -4 & -7 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

3. The solution set is the line $x = 3 + 4t, y = 1 + 4t, z = 5$.

   Solution: Let $y = 1+4t = s$. Then $4t = s-1 \Rightarrow x = 3+4t = 3+s-1 = 2+s$. Thus, the given line can also be represented by the parametric equation $x = 2 + s$, $y = s$, $z = 5$. So the variable y is free and the corresponding reduced row-echelon matrix would be

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

4. The solution set is the plane $3x - 4y + 6z = 9$.

   Solution: Choose a parametrization of the given equation of the plane. Let $y = s$, $z = t$. Then $3x = 9 + 4s - 6t \Rightarrow x = 3 + (4/3)s - 2t$. Thus, the corresponding reduced row-echelon matrix would be

$$\left[\begin{array}{ccc|c} 1 & -4/3 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

6. Matrix Arithmetic

Problem 1: Given the matrices

$$E = \begin{bmatrix} -1 & 3 & 5 \\ -4 & 2 & 2 \end{bmatrix} \text{ and } F = \begin{bmatrix} 3 & -5 & 1 \\ 6 & -2 & -1 \end{bmatrix}$$

compute $E + F$, $3E$ and $4E - 2F$.

Solution:

$$E + F = \begin{bmatrix} 2 & -2 & 6 \\ 2 & 0 & 1 \end{bmatrix}$$

$$3E = \begin{bmatrix} -3 & 9 & 15 \\ -12 & 6 & 6 \end{bmatrix}$$

$$4E - 2F = \begin{bmatrix} -10 & 22 & 18 \\ -28 & 12 & 10 \end{bmatrix}$$

Problem 2: Given the matrices

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \quad , \quad B = \begin{bmatrix} -1 & 3 & 5 \\ -4 & 2 & 2 \end{bmatrix} \quad , \quad C = \begin{bmatrix} 3 & -5 \\ 6 & -2 \\ 0 & 4 \end{bmatrix} \quad , \quad D = \begin{bmatrix} 2 & 3 & -2 \\ -1 & 2 & 3 \\ 0 & 2 & 4 \end{bmatrix}$$

compute all possible produces of pairs of matrices.

Solution: The possible products are A^2, AB, BC, BD, CA, CB, DC and D^2.

$$A^2 = \begin{bmatrix} 3 & 5 \\ -5 & 8 \end{bmatrix}$$

$$AB = \begin{bmatrix} -6 & 8 & 12 \\ -11 & 3 & 1 \end{bmatrix}$$

$$BC = \begin{bmatrix} 15 & 19 \\ 0 & 24 \end{bmatrix}$$

$$BD = \begin{bmatrix} -5 & 13 & 31 \\ -10 & -4 & 22 \end{bmatrix}$$

$$CA = \begin{bmatrix} 11 & -12 \\ 14 & 0 \\ -4 & 12 \end{bmatrix}$$

$$CB = \begin{bmatrix} 17 & -1 & 5 \\ 2 & 14 & 26 \\ -16 & 8 & 8 \end{bmatrix}$$

$$DC = \begin{bmatrix} 24 & -24 \\ 9 & 13 \\ 12 & 12 \end{bmatrix}$$

$$D^2 = \begin{bmatrix} 1 & 8 & -3 \\ -4 & 7 & 20 \\ -2 & 12 & 22 \end{bmatrix}$$

Problem 3: Given the matrices A and D from the previous problem compute $A^2 - 5A$ and $D^3 + 2D$.

Solution: As computed before $A^2 = \begin{bmatrix} 3 & 5 \\ -5 & 8 \end{bmatrix}$. Thus,

$$A^2 - 5A = \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix}$$

As computed before $D^2 = \begin{bmatrix} 1 & 8 & -3 \\ -4 & 7 & 20 \\ -2 & 12 & 22 \end{bmatrix}$. Then we have

$$D^3 = D^2.D = \begin{bmatrix} -6 & 13 & 10 \\ -15 & 42 & 109 \\ -16 & 62 & 126 \end{bmatrix}$$

Thus,

$$D^3 + 2D = \begin{bmatrix} -2 & 19 & 6 \\ -17 & 46 & 115 \\ -16 & 66 & 134 \end{bmatrix}$$

7. The Multiplicative Identity and Inverse of Matrices

Problem 1: Determine whether each matrix is invertible or not and find the inverse matrix for each invertible matrix. Check your answer by multiplying the matrices.

1. $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

   Solution: $[A|I_2] = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{array}\right] \xrightarrow{R_2 \to -R_2} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{array}\right]$
   $\xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 \end{array}\right]$. Thus, $A^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$

2. $\begin{bmatrix} -1 & 2 \\ 1 & 5 \end{bmatrix}$

   Solution: $[A|I_2] = \left[\begin{array}{cc|cc} -1 & 2 & 1 & 0 \\ 1 & 5 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to -R_1} \left[\begin{array}{cc|cc} 1 & -2 & -1 & 0 \\ 1 & 5 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cc|cc} 1 & -2 & -1 & 0 \\ 0 & 7 & 1 & 1 \end{array}\right]$
   $\xrightarrow{R_2 \to (1/7)R_2} \left[\begin{array}{cc|cc} 1 & -2 & -1 & 0 \\ 0 & 1 & 1/7 & 1/7 \end{array}\right] \xrightarrow{R_1 \to R_1 + 2R_2} \left[\begin{array}{cc|cc} 1 & 0 & -5/7 & 2/7 \\ 0 & 1 & 1/7 & 1/7 \end{array}\right]$
   Thus, $A^{-1} = \begin{bmatrix} -5/7 & 2/7 \\ 1/7 & 1/7 \end{bmatrix}$

3. $\begin{bmatrix} 3 & -4 \\ 6 & 8 \end{bmatrix}$

   Solution: $[A|I_2] = \left[\begin{array}{cc|cc} 3 & -4 & 1 & 0 \\ 6 & 8 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to (1/3)R_1} \left[\begin{array}{cc|cc} 1 & -4/3 & 1/3 & 0 \\ 6 & 8 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - 6R_1} \left[\begin{array}{cc|cc} 1 & -4/3 & 1/3 & 0 \\ 0 & 16 & -2 & 1 \end{array}\right]$
   $\xrightarrow{R_2 \to (1/16)R_2} \left[\begin{array}{cc|cc} 1 & -4/3 & 1/3 & 0 \\ 0 & 1 & -1/8 & 1/16 \end{array}\right] \xrightarrow{R_1 \to R_1 + (4/3)R_2} \left[\begin{array}{cc|cc} 1 & 0 & 1/6 & 1/12 \\ 0 & 1 & -1/8 & 1/16 \end{array}\right]$
   Thus, $A^{-1} = \begin{bmatrix} 1/6 & 1/12 \\ -1/8 & 1/16 \end{bmatrix}$

4. $\begin{bmatrix} 1/2 & -1/3 \\ 1/4 & 1/2 \end{bmatrix}$

   Solution: $[A|I_2] = \left[\begin{array}{cc|cc} 1/2 & -1/3 & 1 & 0 \\ 1/4 & 1/2 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to 2R_1} \left[\begin{array}{cc|cc} 1 & -2/3 & 2 & 0 \\ 1/4 & 1/2 & 0 & 1 \end{array}\right]$
   $\xrightarrow{R_2 \to R_2 - (1/4)R_1} \left[\begin{array}{cc|cc} 1 & -2/3 & 2 & 0 \\ 0 & 2/3 & -1/2 & 1 \end{array}\right] \xrightarrow{R_2 \to (3/2)R_2} \left[\begin{array}{cc|cc} 1 & -2/3 & 2 & 0 \\ 0 & 1 & -3/4 & 3/2 \end{array}\right]$
   $\xrightarrow{R_1 \to R_1 + (2/3)R_2} \left[\begin{array}{cc|cc} 1 & 0 & 3/2 & 1 \\ 0 & 1 & -3/4 & 3/2 \end{array}\right]$ Thus, $A^{-1} = \begin{bmatrix} 3/2 & 1 \\ -3/4 & 3/2 \end{bmatrix}$

Problem 2: Determine whether each matrix is invertible or not and find the inverse matrix for each invertible matrix. Check your answer by multiplying the matrices.

1. $\begin{bmatrix} 1 & 1 & -2 \\ 1 & 2 & 3 \\ 3 & 5 & 6 \end{bmatrix}$

Solution: $[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 5 & -1 & 1 & 0 \\ 0 & 2 & 12 & -3 & 0 & 1 \end{array}\right]$

$\xrightarrow[R_3 \to R_3 - 2R_2]{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -7 & 2 & -1 & 0 \\ 0 & 1 & 5 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & -2 & 1 \end{array}\right] \xrightarrow{R_3 \to (1/2)R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -7 & 2 & -1 & 0 \\ 0 & 1 & 5 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1/2 & -1 & 1/2 \end{array}\right]$

$\xrightarrow[R_2 \to R_2 - 5R_3]{R_1 \to R_1 + 7R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3/2 & -8 & 7/2 \\ 0 & 1 & 0 & 3/2 & 6 & -5/2 \\ 0 & 0 & 1 & -1/2 & -1 & 1/2 \end{array}\right]$. Thus, $A^{-1} = \begin{bmatrix} -3/2 & -8 & 7/2 \\ 3/2 & 6 & -5/2 \\ -1/2 & -1 & 1/2 \end{bmatrix}$

2. $\begin{bmatrix} -2 & 1 & 3 \\ 3 & -1 & 5 \\ 12 & -5 & 1 \end{bmatrix}$

Solution: $[A|I_3] = \left[\begin{array}{ccc|ccc} -2 & 1 & 3 & 1 & 0 & 0 \\ 3 & -1 & 5 & 0 & 1 & 0 \\ 12 & -5 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to (-1/2)R_1} \left[\begin{array}{ccc|ccc} 1 & -1/2 & -3/2 & -1/2 & 0 & 0 \\ 3 & -1 & 5 & 0 & 1 & 0 \\ 12 & -5 & 1 & 0 & 0 & 1 \end{array}\right]$

$\xrightarrow[R_3 \to R_3 - 12R_1]{R_2 \to R_2 - 3R_1} \left[\begin{array}{ccc|ccc} 1 & -1/2 & -3/2 & -1/2 & 0 & 0 \\ 0 & 1/2 & 19/2 & 3/2 & 1 & 0 \\ 0 & 1 & 19 & 6 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to 2R_2} \left[\begin{array}{ccc|ccc} 1 & -1/2 & -3/2 & -1/2 & 0 & 0 \\ 0 & 1 & 19 & 3 & 2 & 0 \\ 0 & 1 & 19 & 6 & 0 & 1 \end{array}\right]$

$\xrightarrow[R_3 \to R_3 - R_2]{R_1 \to R_1 + (1/2)R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 8 & 1 & 1 & 0 \\ 0 & 1 & 19 & 3 & 2 & 0 \\ 0 & 0 & 0 & 3 & -2 & 1 \end{array}\right]$

Since we have an all zero row, the given matrix is singular.

3. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 8 \\ 2 & 5 & 3 \end{bmatrix}$

Solution: $[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 0 & 8 & 0 & 1 & 0 \\ 2 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 8 & 0 & 1 & 0 \\ 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 0 & 1 \end{array}\right]$

$\xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 8 & 0 & 1 & 0 \\ 0 & 2 & -5 & 1 & -1 & 0 \\ 0 & 5 & -13 & 0 & -2 & 1 \end{array}\right] \xrightarrow{R_2 \to (1/2)R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 8 & 0 & 1 & 0 \\ 0 & 1 & -5/2 & 1/2 & -1/2 & 0 \\ 0 & 5 & -13 & 0 & -2 & 1 \end{array}\right]$

$\xrightarrow{R_3 \to R_3 - 5R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 8 & 0 & 1 & 0 \\ 0 & 1 & -5/2 & 1/2 & -1/2 & 0 \\ 0 & 0 & -1/2 & -5/2 & 1/2 & 1 \end{array}\right] \xrightarrow{R_3 \to -2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 8 & 0 & 1 & 0 \\ 0 & 1 & -5/2 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 5 & -1 & -2 \end{array}\right]$

$\xrightarrow[R_2 \to R_2 + (5/2)R_3]{R_1 \to R_1 - 8R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 9 & 16 \\ 0 & 1 & 0 & 13 & -3 & -5 \\ 0 & 0 & 1 & 5 & -1 & -2 \end{array}\right]$. Thus, $A^{-1} = \begin{bmatrix} -40 & 9 & 16 \\ 13 & -3 & -5 \\ 5 & -1 & -2 \end{bmatrix}$

4. $\begin{bmatrix} 1/3 & 1/4 & 0 \\ -1/3 & 1/4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Solution: $[A|I_3] = \left[\begin{array}{ccc|ccc} 1/3 & 1/4 & 0 & 1 & 0 & 0 \\ -1/3 & 1/4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to 3R_1} \left[\begin{array}{ccc|ccc} 1 & 3/4 & 0 & 3 & 0 & 0 \\ -1/3 & 1/4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right]$

$\xrightarrow{R_2 \to R_2 + (1/3)R_1} \left[\begin{array}{ccc|ccc} 1 & 3/4 & 0 & 3 & 0 & 0 \\ 0 & 1/2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to 2R_2} \left[\begin{array}{ccc|ccc} 1 & 3/4 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 2 & 2 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right]$

$\xrightarrow{R_1 \to R_1 - (3/4)R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & -3/2 & 0 \\ 0 & 1 & 0 & 2 & 2 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to (1/3)R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & -3/2 & 0 \\ 0 & 1 & 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/3 \end{array}\right]$

Thus, $A^{-1} = \begin{bmatrix} 3/2 & -3/2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$

Problem 3: Use an inverse matrix from above to solve the system of linear equations

$$\begin{gathered} -x + 2y = 3 \\ x + 5y = -1 \end{gathered}$$

Solution: The given system is $AX = b$ where $A = \begin{bmatrix} -1 & 2 \\ 1 & 5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

As computer earlier $A^{-1} = \begin{bmatrix} -5/7 & 2/7 \\ 1/7 & 1/7 \end{bmatrix}$ and we have $X = A^{-1}b$

Thus, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5/7 & 2/7 \\ 1/7 & 1/7 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -17/7 \\ 2/7 \end{bmatrix} \Rightarrow x = -17/7,\ y = 2/7.$

Problem 4: Use an inverse matrix from above to solve the system of linear equations

$$\begin{gathered} x + y - 2z = -2 \\ x + 2y + 3z = 5 \\ 3x + 5y + 6z = 3 \end{gathered}$$

Solution: The given system is $AX = b$ where $A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 2 & 3 \\ 3 & 5 & 6 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $b = \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}$.

As computed earlier, $A^{-1} = \begin{bmatrix} -3/2 & -8 & 7/2 \\ 3/2 & 6 & -5/2 \\ -1/2 & -1 & 1/2 \end{bmatrix}$ and we have $X = A^{-1}b$

Thus, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3/2 & -8 & 7/2 \\ 3/2 & 6 & -5/2 \\ -1/2 & -1 & 1/2 \end{bmatrix} \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} -53/2 \\ 39/2 \\ -5/2 \end{bmatrix} \Rightarrow x = -53/2,\ y = 39/2,\ z = -5/2.$

Problem 5: Use an inverse matrix from above to solve the system of linear equations

$$\begin{array}{rrrcr} (1/3)x & +(1/4)y & & = & 8 \\ (-1/3)x & +(1/4)y & & = & -3 \\ & & 3z & = & 4 \end{array}$$

Solution: The given system is $AX = b$ where $A = \begin{bmatrix} 1/3 & 1/4 & 0 \\ -1/3 & 1/4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $b = \begin{bmatrix} 8 \\ -3 \\ 4 \end{bmatrix}$.

As computed earlier, $A^{-1} = \begin{bmatrix} 3/2 & -3/2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$ and we have $X = A^{-1}b$

Thus, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3/2 & -3/2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 8 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 33/2 \\ 10 \\ 4/3 \end{bmatrix} \Rightarrow x = 33/2,\ y = 10,\ z = 4/3.$

8. Determinants

Problem 1: Compute the determinants of the matrices

1. $\begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix}$

   Solution: $\begin{vmatrix} 3 & -2 \\ 1 & 4 \end{vmatrix} = 3(4) - 1(-2) = 14$

2. $\begin{bmatrix} -4 & 6 \\ 2 & -3 \end{bmatrix}$

   Solution: $\begin{vmatrix} -4 & 6 \\ 2 & -3 \end{vmatrix} = -4(-3) - 2(6) = 12 - 12 = 0$

Problem 2: Compute the determinants of the matrices in two ways, first by expanding along a row and then by expanding down a column.

1. $\begin{bmatrix} 3 & -2 & -1 \\ 1 & -4 & 2 \\ 3 & 0 & -5 \end{bmatrix}$

   Solution: Expanding by the third row we have

$$\begin{vmatrix} 3 & -2 & -1 \\ 1 & -4 & 2 \\ 3 & 0 & -5 \end{vmatrix} = 3\begin{vmatrix} -2 & -1 \\ -4 & 2 \end{vmatrix} - 0\begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} + (-5)\begin{vmatrix} 3 & -2 \\ 1 & -4 \end{vmatrix} = 3(-8) - 0 - 5(-10) = 26$$

   Expanding by the second column we have

$$\begin{vmatrix} 3 & -2 & -1 \\ 1 & -4 & 2 \\ 3 & 0 & -5 \end{vmatrix} = -(-2)\begin{vmatrix} 1 & 2 \\ 3 & -5 \end{vmatrix} + (-4)\begin{vmatrix} 3 & -1 \\ 3 & -5 \end{vmatrix} - 0\begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} = 2(-11) - 4(-12) - 0 = 26$$

2. $\begin{bmatrix} -2 & 3 & 1 \\ 5 & -2 & 5 \\ 1 & 4 & 7 \end{bmatrix}$

   Solution: Expanding by the first row we have

$$\begin{vmatrix} -2 & 3 & 1 \\ 5 & -2 & 5 \\ 1 & 4 & 7 \end{vmatrix} = -2\begin{vmatrix} -2 & 5 \\ 4 & 7 \end{vmatrix} - 3\begin{vmatrix} 5 & 5 \\ 1 & 7 \end{vmatrix} + 1\begin{vmatrix} 5 & -2 \\ 1 & 4 \end{vmatrix} = -2(-34) - 3(30) + 1(22) = 0$$

   Expanding by the first column we have

$$\begin{vmatrix} -2 & 3 & 1 \\ 5 & -2 & 5 \\ 1 & 4 & 7 \end{vmatrix} = -2\begin{vmatrix} -2 & 5 \\ 4 & 7 \end{vmatrix} - 5\begin{vmatrix} 3 & 1 \\ 4 & 7 \end{vmatrix} + 1\begin{vmatrix} 3 & 1 \\ -2 & 5 \end{vmatrix} = -2(-34) - 5(17) + 1(17) = 0$$

Problem 3: Compute the determinants of the matrices

1. $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

   Solution: Expanding along first row or fist column we have

$$\begin{vmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -2 \end{vmatrix} = 3 \begin{vmatrix} -4 & 0 \\ 0 & -2 \end{vmatrix} = 3(-4 \times -2) = 24$$

2. $\begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix}$

   Solution: Expanding along first row or first column we have

$$\begin{vmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{vmatrix} = A_{11} \begin{vmatrix} A_{22} & 0 \\ 0 & A_{33} \end{vmatrix} = A_{11}A_{22}A_{33}$$

Problem 4: Compute the determinants of the matrices

1. $\begin{bmatrix} 3 & -2 & 1 \\ 0 & -4 & 9 \\ 0 & 0 & -2 \end{bmatrix}$

   Solution: Expanding along first column we have

$$\begin{vmatrix} 3 & -2 & 1 \\ 0 & -4 & 9 \\ 0 & 0 & -2 \end{vmatrix} = 3 \begin{vmatrix} -4 & 9 \\ 0 & -2 \end{vmatrix} = 3(8 - 0) = 24$$

2. $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix}$

   Solution: Expanding along first column we have

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{vmatrix} = A_{11} \begin{vmatrix} A_{22} & A_{23} \\ 0 & A_{33} \end{vmatrix} = A_{11}A_{22}A_{33}$$

Problem 5: Compute the determinants of the matrices

1. $\begin{bmatrix} 3 & 0 & 0 \\ 5 & -4 & 0 \\ 7 & -9 & -2 \end{bmatrix}$

Solution: Expanding along first row we have

$$\begin{vmatrix} 3 & 0 & 0 \\ 5 & -4 & 0 \\ 7 & -9 & -2 \end{vmatrix} = 3 \begin{vmatrix} -4 & 0 \\ -9 & -2 \end{vmatrix} = 3(8-0) = 24$$

2. $\begin{bmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

Solution: Expanding along the first row we have

$$\begin{vmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = A_{11} \begin{vmatrix} A_{22} & 0 \\ A_{32} & A_{33} \end{vmatrix} = A_{11}A_{22}A_{33}$$

Problem 6: Compute the determinant of the matrix in two ways, first by expanding along a row and then by expanding down a column.

$$\begin{bmatrix} 2 & -3 & 1 & 5 \\ -1 & 4 & 0 & 2 \\ 0 & 2 & -2 & 1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$$

Solution: Expanding along the fourth row we have

$$\begin{vmatrix} 2 & -3 & 1 & 5 \\ -1 & 4 & 0 & 2 \\ 0 & 2 & -2 & 1 \\ 3 & 1 & 0 & 1 \end{vmatrix} = -3 \begin{vmatrix} -3 & 1 & 5 \\ 4 & 0 & 2 \\ 2 & -2 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 2 \\ 0 & -2 & 1 \end{vmatrix} - 0 \begin{vmatrix} 2 & -3 & 5 \\ -1 & 4 & 2 \\ 0 & 2 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & -3 & 1 \\ -1 & 4 & 0 \\ 0 & 2 & -2 \end{vmatrix}$$

$$\begin{vmatrix} -3 & 1 & 5 \\ 4 & 0 & 2 \\ 2 & -2 & 1 \end{vmatrix} = \begin{vmatrix} -3 & 1 & 5 \\ 4 & 0 & 2 \\ -4 & 0 & 11 \end{vmatrix} \ (R_3 \to R_3 + 2R_1) \ \Rightarrow \begin{vmatrix} -3 & 1 & 5 \\ 4 & 0 & 2 \\ 2 & -2 & 1 \end{vmatrix} = -1 \begin{vmatrix} 4 & 2 \\ -4 & 11 \end{vmatrix} = -52$$

$$\begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 2 \\ 0 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 11 & 5 \\ -1 & 4 & 2 \\ 0 & 0 & 1 \end{vmatrix} \ (C_2 \to C_2 + 2C_3) \ \Rightarrow \begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 2 \\ 0 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 11 \\ -1 & 4 \end{vmatrix} = 19$$

$$\begin{vmatrix} 2 & -3 & 1 \\ -1 & 4 & 0 \\ 0 & 2 & -2 \end{vmatrix} = \begin{vmatrix} 2 & -3 & 1 \\ -1 & 4 & 0 \\ 4 & -4 & 0 \end{vmatrix} \ (R_3 \to R_3 + 2R_1) \ \Rightarrow \begin{vmatrix} 2 & -3 & 1 \\ -1 & 4 & 0 \\ 0 & 2 & -2 \end{vmatrix} = \begin{vmatrix} -1 & 4 \\ 4 & -4 \end{vmatrix} = -12$$

$$\text{Thus, } \begin{vmatrix} 2 & -3 & 1 & 5 \\ -1 & 4 & 0 & 2 \\ 0 & 2 & -2 & 1 \\ 3 & 1 & 0 & 1 \end{vmatrix} = -3(-52) + 1(19) + 1(-12) = 163$$

Expanding along the third column we have

$$\begin{vmatrix} 2 & -3 & 1 & 5 \\ -1 & 4 & 0 & 2 \\ 0 & 2 & -2 & 1 \\ 3 & 1 & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} -1 & 4 & 2 \\ 0 & 2 & 1 \\ 3 & 1 & 1 \end{vmatrix} - 0\begin{vmatrix} 2 & -3 & 5 \\ 0 & 2 & 1 \\ 3 & 1 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & -3 & 5 \\ -1 & 4 & 2 \\ 3 & 1 & 1 \end{vmatrix} - 0\begin{vmatrix} 2 & -3 & 5 \\ -1 & 4 & 2 \\ 0 & 2 & 1 \end{vmatrix}$$

$$\begin{vmatrix} -1 & 4 & 2 \\ 0 & 2 & 1 \\ 3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 4 & 2 \\ 0 & 2 & 1 \\ 0 & 13 & 7 \end{vmatrix} \ (R_3 \to R_3 + 3R_1) \ \Rightarrow \begin{vmatrix} -1 & 4 & 2 \\ 0 & 2 & 1 \\ 3 & 1 & 1 \end{vmatrix} = -1\begin{vmatrix} 2 & 1 \\ 13 & 7 \end{vmatrix} = -1$$

$$\begin{vmatrix} 2 & -3 & 5 \\ -1 & 4 & 2 \\ 3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 5 & 9 \\ -1 & 4 & 2 \\ 0 & 13 & 7 \end{vmatrix} \ \begin{matrix} (R_1 \to R_1 + 2R_2) \\ (R_3 \to R_3 + 3R_2) \end{matrix} \ \Rightarrow \begin{vmatrix} 2 & -3 & 5 \\ -1 & 4 & 2 \\ 3 & 1 & 1 \end{vmatrix} = -(-1)\begin{vmatrix} 5 & 9 \\ 13 & 7 \end{vmatrix} = -82$$

$$\text{Thus, } \begin{vmatrix} 2 & -3 & 1 & 5 \\ -1 & 4 & 0 & 2 \\ 0 & 2 & -2 & 1 \\ 3 & 1 & 0 & 1 \end{vmatrix} = 1(-1) - 2(-82) = 163$$

Problem 7: Given a 2×2 matrix A with a nonzero determinant, what is the determinant of its inverse matrix A^{-1}?

Solution: Using the result of problem 8 below, we have $\det(AA^{-1}) = \det(A)\det(A^{-1})$

But $AA^{-1} = I_2 \Rightarrow \det(A)\det(A^{-1}) = \det(I_2) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$. Therefore, $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

ALTERNATIVELY, let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

$\Rightarrow \det(A^{-1}) = \dfrac{1}{(ad-bc)^2}\begin{vmatrix} d & -b \\ -c & a \end{vmatrix} = \dfrac{ad-bc}{(ad-bc)^2} = \dfrac{1}{ad-bc} = \dfrac{1}{\det(A)}$.

Problem 8: Given two 2×2 matrices A and B show that $\det(AB) = \det A \det B$

Solution: Let $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$.

Then $AB = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{bmatrix}$.

$\Rightarrow \det(AB) = (a_1a_2 + b_1c_2)(c_1b_2 + d_1d_2) - (a_1b_2 + b_1d_2)(c_1a_2 + d_1c_2)$
$= \cancel{a_1c_1a_2b_2} + a_1d_1a_2d_2 + b_1c_1b_2c_2 + \cancel{b_1d_1c_2d_2} - \cancel{a_1c_1a_2b_2} - a_1d_1b_2c_2 - b_1c_1a_2d_2 - \cancel{b_1d_1c_2d_2}$
$= a_1d_1(a_2d_2 - b_2c_2) - b_1c_1(a_2d_2 - b_2c_2) = (a_1d_1 - b_1c_1)(a_2d_2 - b_2c_2) = \det(A)\det(B)$

Problem 9: Suppose A is a 3×3 matrix and $\det A = 5$. Using the properties of 3×3 determinants find the determinant of:-

1. a matrix obtained by interchanging two rows of A,
2. a matrix obtained by multiplying a row by 3,
3. a matrix obtained by multiplying a row by -2,
4. the matrix $3A$,
5. the matrix $(-2)A$,

6. the matrix obtained by adding the second row of A to the first row of A,
7. the transpose of A, A^T.

Solution:

1. $-\det A = -5$
2. $3 \det A = 15$
3. $-2 \det A = -10$
4. $3^3 \det A = 135$
5. $(-2)^3 \det A = -40$
6. $\det A = 5$
7. $\det A^T = \det A = 5$

Problem 10: Given two 3×3 matrices A and B show that $\det(AB) = \det A \det B$.

Solution: Each row operation is equivalent to left multiplying with some matrix, called an elementary matrix. The following are the elementary matrices corresponding to each row operation.

$$E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} (R_1 \leftrightarrow R_2),\ E_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} (R_2 \leftrightarrow R_3),\ E_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} (R_3 \leftrightarrow R_1)$$

$$E_1^\alpha = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} (R_1 \to \alpha R_1),\ E_2^\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} (R_2 \to \alpha R_2),\ E_3^\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{bmatrix} (R_3 \to \alpha R_3),$$

$$E_{12}^\alpha = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} (R_1 \to R_1 + \alpha R_2),\ E_{21}^\alpha = \begin{bmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} (R_1 \to R_2 + \alpha R_1)$$

$$E_{23}^\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{bmatrix} (R_2 \to R_2 + \alpha R_3),\ E_{32}^\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \alpha & 1 \end{bmatrix} (R_3 \to R_3 + \alpha R_2)$$

$$E_{13}^\alpha = \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} (R_1 \to R_1 + \alpha R_3),\ E_{31}^\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & 0 & 1 \end{bmatrix} (R_3 \to R_3 + \alpha R_1)$$

Note that $E_{ij}^{-1} = E_{ij}$, $(E_i^\alpha)^{-1} = E_i^{1/\alpha}$, $(E_{ij}^\alpha)^{-1} = E_{ij}^{-\alpha}$
and that $\det(E_{ij}) = -1$, $\det(E_i^\alpha) = \alpha$, $\det(E_{ij}^\alpha) = 1$.

Also note that if E is any elementary matrix then $\det(EB) = \det(E)\det(B)$.

Now let E_A be the reduced row echelon form of A obtained by some s number of elementary row operations then $E_A = E_s E_{s-1} \cdots E_2 E_1 A$ where E_i, $1 \leq i \leq s$ are elementary matrices.

So we have $A = E_1^{-1}E_2^{-1}\cdots E_s^{-1}E_A$ where E_i^{-1}, $1 \le i \le s$ are again elementary matrices.

Thus, $\det(AB) = \det(E_1^{-1}\cdots E_s^{-1}E_AB) = \det(E_1^{-1})\cdots\det(E_s^{-1})\det(E_AB)$

and $\det(A) = \det(E_1^{-1})\cdots\det(E_s^{-1})\det(E_A)$.

Now, if E_A has a zero row then so does E_AB, in which case $\det(E_A) = 0 = \det(E_AB)$ and we have $\det(AB) = 0 = \det(A) \Rightarrow \det(AB) = \det(A)\det(B) = 0$.

Else, if E_A does not have a zero row, then $E_A = I_3 \Rightarrow E_AB = I_3B = B$ and we have $\det(AB) = \det(E_1^{-1})\cdots\det(E_s^{-1})\det(B)$. Also, $\det(A) = \det(E_1^{-1})\cdots\det(E_s^{-1})\det(I_3)$.

Since $\det(I_3) = 1$, we have $\det(AB) = \det(A)\det(B)$.

(The same proof also works for $n \times n$ matrices for any n)

Problem 11: Given a 3×3 matrix A with a nonzero determinant, show that the determinant of its inverse matrix A^{-1} is $1/\det A$.

Solution: $AA^{-1} = I_3 \Rightarrow \det(AA^{-1}) = \det(I_3) = 1$. Using the result of previous problem we have $\det(A)\det(A^{-1}) = 1$. Since $\det(A)$ is nonzero, we can divide both sides by $\det(A)$. Therefore, $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Problem 12: Given a 3×3 matrix A with a nonzero determinant, define the 3×3 matrix C by $C_{ij} = (-1)^{i+j}\det A(i|j)$ and show that $AC^T = (\det A)I_3$.

Solution: The diagonal entries $(AC^T)_{ii}$ of AC^T for $1 \le i \le 3$ are given by

$$(AC^T)_{ii} = A_{i1}(C^T)_{1i} + A_{i2}(C^T)_{2i} + A_{i3}(C^T)_{3i} = A_{i1}C_{i1} + A_{i2}C_{i2} + A_{i3}C_{i3}$$

$$\Rightarrow (AC^T)_{ii} = A_{i1}(-1)^{i+1}\det A(i|1) + A_{i2}(-1)^{i+2}\det A(i|2) + A_{i3}(-1)^{i+3}\det A(i|3)$$

which is the formula of $\det(A)$ when expanding by the i^{th} row. Thus, $(AC^T)_{ii} = \det(A)$.

Now if we look at the off-diagonal entry $(AC^T)_{ij}$ for $i \ne j$. Then we have

$$(AC^T)_{ij} = A_{i1}(-1)^{j+1}\det A(j|1) + A_{i2}(-1)^{j+2}\det A(j|2) + A_{i3}(-1)^{j+3}\det A(j|3)$$

which is the formula (when expanding by j^{th} row) for determinant of a matrix whose i^{th} and j^{th} row are the same and we know that determinant of such a matrix is 0.

Thus, $AC^T = \begin{bmatrix} \det(A) & 0 & 0 \\ 0 & \det(A) & 0 \\ 0 & 0 & \det(A) \end{bmatrix} = \det(A)I_3$.

9. Matrices as Functions

Problem 1: Carry out the computations on both sides of the equation to see that they are equal.

$$\begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 5 \end{bmatrix} \right) = \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \\ 5 \end{bmatrix}$$

Solution:

$$\text{LHS} = \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 5 \end{bmatrix} \right) = \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -8 \\ 19 \end{bmatrix}$$

$$\text{RHS} = \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 11 \\ 10 \\ -10 \end{bmatrix} + \begin{bmatrix} -14 \\ -18 \\ 29 \end{bmatrix} = \begin{bmatrix} -3 \\ -8 \\ 19 \end{bmatrix}$$

Problem 2: Carry out the computations on both sides of the equation to see that they are equal.

$$\begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \left(5 \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \right) = 5 \left(\begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \right)$$

Solution:

$$\text{LHS} = \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \left(5 \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \right) = \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 10 \\ -15 \end{bmatrix} = \begin{bmatrix} 55 \\ 50 \\ -50 \end{bmatrix}$$

$$\text{RHS} = 5 \left(\begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \right) = 5 \begin{bmatrix} 11 \\ 10 \\ -10 \end{bmatrix} = \begin{bmatrix} 55 \\ 50 \\ -50 \end{bmatrix}$$

Problem 3: Given the matrix $A = \begin{bmatrix} 3 & 1 \\ 4 & 0 \\ -5 & 2 \end{bmatrix}$ determine which of the following vectors are in the image of A:-

$$\begin{bmatrix} 4 \\ 4 \\ -3 \end{bmatrix} \quad \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix} \quad \begin{bmatrix} -5 \\ -8 \\ 12 \end{bmatrix}$$

Solution: Form the augmented matrix $[A|v_1\ v_2\ v_3]$ and bring A in reduced row-echelon form.

$$\left[\begin{array}{cc|ccc} 3 & 1 & 4 & 5 & -5 \\ 4 & 0 & 4 & 3 & -8 \\ -5 & 2 & -3 & 2 & 12 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|ccc} 4 & 0 & 4 & 3 & -8 \\ 3 & 1 & 4 & 5 & -5 \\ -5 & 2 & -3 & 2 & 12 \end{array}\right] \xrightarrow{R_2 \to (1/4)R_2} \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 3/4 & -2 \\ 3 & 1 & 4 & 5 & -5 \\ -5 & 2 & -3 & 2 & 12 \end{array}\right]$$

$$\xrightarrow[R_3\to R_3+5R_1]{R_2\to R_2-3R_1} \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 3/4 & -2 \\ 0 & 1 & 1 & 11/4 & 1 \\ 0 & 2 & 2 & 23/4 & 2 \end{array}\right] \xrightarrow{R_3\to R_3-2R_2} \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 3/4 & -2 \\ 0 & 1 & 1 & 11/4 & 1 \\ 0 & 0 & 0 & 1/4 & 0 \end{array}\right]$$

We see that for the first and third vectors there are infinitely many solutions while the for the second vector there are no solutions. Therefore, the $\boxed{\text{first and third vector}}$ lie in the image of A but the second vector does not.

Problem 4: Given the matrix $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & -2 \\ -3 & 3 & 1 \end{bmatrix}$ determine which of the following vectors are in the image of A:-

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -4 \\ 2 \\ 3 \end{bmatrix} \quad \begin{bmatrix} -5 \\ -4 \\ -3 \end{bmatrix}$$

Solution: Form the augmented matrix $[A|v_1\ v_2\ v_3]$ and bring A in reduced row-echelon form.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & -4 & -5 \\ 2 & -1 & -2 & 1 & 2 & -4 \\ -3 & 3 & 1 & 0 & 3 & -3 \end{array}\right] \xrightarrow[R_3\to R_3+3R_1]{R_2\to R_2-2R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & -4 & -5 \\ 0 & 1 & -2 & -1 & 10 & 6 \\ 0 & 0 & 1 & 3 & -9 & -18 \end{array}\right]$$

$$\xrightarrow{R_1\to R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & 6 & 1 \\ 0 & 1 & -2 & -1 & 10 & 6 \\ 0 & 0 & 1 & 3 & -9 & -18 \end{array}\right] \xrightarrow[R_2\to R_2+2R_3]{R_1\to R_1+2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & -12 & -35 \\ 0 & 1 & 0 & 5 & -8 & -30 \\ 0 & 0 & 1 & 3 & -9 & -18 \end{array}\right]$$

We see that for all the three given vectors there is a unique solution. Therefore, $\boxed{\text{all the three vectors}}$ lie in the image of A.

Problem 5: In the previous problem use the inverse of A to find the vectors in $\mathbb{R}^3$ that are mapped by A to the three vectors listed.

Solution: Find the inverse of A.

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 2 & -1 & -2 & 0 & 1 & 0 \\ -3 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3\to R_3+3R_1]{R_2\to R_2-2R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1\to R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & 1 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 & 1 \end{array}\right] \xrightarrow[R_2\to R_2+2R_3]{R_1\to R_1+2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & 1 & 2 \\ 0 & 1 & 0 & 4 & 1 & 2 \\ 0 & 0 & 1 & 3 & 0 & 1 \end{array}\right]$$

Thus, $A^{-1} = \begin{bmatrix} 5 & 1 & 2 \\ 4 & 1 & 2 \\ 3 & 0 & 1 \end{bmatrix}$ and we have:-

$$A^{-1}v_1 = \begin{bmatrix} 5 & 1 & 2 \\ 4 & 1 & 2 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 3 \end{bmatrix}$$

$$A^{-1}v_2 = \begin{bmatrix} 5 & 1 & 2 \\ 4 & 1 & 2 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -12 \\ -8 \\ -9 \end{bmatrix}$$

$$A^{-1}v_3 = \begin{bmatrix} 5 & 1 & 2 \\ 4 & 1 & 2 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ -4 \\ -3 \end{bmatrix} = \begin{bmatrix} -35 \\ -30 \\ -18 \end{bmatrix}$$

Note that these three vector appeared as column vectors to the right of the vertical line in the final augmented matrix of the previous solution.

Problem 6: Given the matrix $B = \begin{bmatrix} 5 & 3 & -2 \\ 2 & 0 & 4 \\ 3 & 1 & 2 \end{bmatrix}$ determine which of the following vectors are in the image of B:-

$$\begin{bmatrix} 9 \\ 6 \\ 7 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Solution: Form the augmented matrix $[A|v_1\ v_2\ v_3]$ and bring A in reduced row-echelon form.

$$\left[\begin{array}{ccc|ccc} 5 & 3 & -2 & 9 & 1 & 1 \\ 2 & 0 & 4 & 6 & 1 & -1 \\ 3 & 1 & 2 & 7 & 1 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 2 & 0 & 4 & 6 & 1 & -1 \\ 5 & 3 & -2 & 9 & 1 & 1 \\ 3 & 1 & 2 & 7 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to (1/2)R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 3 & 1/2 & -1/2 \\ 5 & 3 & -2 & 9 & 1 & 1 \\ 3 & 1 & 2 & 7 & 1 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 5R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 3 & 1/2 & -1/2 \\ 0 & 3 & -12 & -6 & -3/2 & 7/2 \\ 0 & 1 & -4 & -2 & -1/2 & 5/2 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 3 & 1/2 & -1/2 \\ 0 & 1 & -4 & -2 & -1/2 & 5/2 \\ 0 & 3 & -12 & -6 & -3/2 & 7/2 \end{array}\right] \xrightarrow{R_3 \to R_3 - 3R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 3 & 1/2 & -1/2 \\ 0 & 1 & -4 & -2 & -1/2 & 5/2 \\ 0 & 0 & 0 & 0 & 0 & -4 \end{array}\right]$$

So we have infinitely many solutions for first two vectors and no solution for the third vector.

Therefore, the first two vectors are in the image of B while the third vector is not.

Problem 7: Let A be a 3×3 matrix. Show that the image under the function defined by A of a line through the origin is the point $(0, 0, 0)$ or a line through the origin. Find an example for each case.

Solution: Any line passing through origin can be expressed by the equation $\vec{r} = t\vec{v}$ where $\vec{v} \neq \vec{0}$ is the direction vector of the line and $t \in \mathbb{R}$. When A is applied to $\vec{v}$ we may have two cases: Either $\vec{w} = A\vec{v} = \vec{0}$ or $\vec{w} = A\vec{v} \neq \vec{0}$.

Case 1: $\vec{w} = \vec{0} \Rightarrow A\vec{r} = A(t\vec{v}) = tA\vec{v} = t\vec{w} = \vec{0}$. Thus, the entire line is mapped to $(0, 0, 0)$.

Case 2: $\vec{w} \neq \vec{0} \Rightarrow A\vec{r} = A(t\vec{v}) = tA\vec{v} = t\vec{w}$. Thus, any vector on the line through $\vec{0}$ and $\vec{v}$ is mapped to a unique vector on the line through $\vec{0}$ and $\vec{w}$. Further, suppose we have some vector on the line through $\vec{0}$ and $\vec{w}$, then it must be $s\vec{w}$ for some $s \in \mathbb{R}$. Then we have a vector $s\vec{v}$ on the line through $\vec{0}$ and $\vec{v}$ which is mapped by A to $s\vec{w}$, since $A(s\vec{v}) = sA\vec{v} = s\vec{w}$. Thus, the image of A in this case is a line through the origin.

Problem 8: Let A and B be 3×3 matrices. Each defines a function from $\mathbb{R}^3$ to $\mathbb{R}^3$. Show that the matrix BA defines the function obtained by composing the function defined by B with the function defined by A (first apply A then B). What happens when the order of applying the functions is reversed?

Solution: We have $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T_A(x) = Ax$ and $T_B : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T_B(x) = Bx$. Now $T_{BA}(x) = (BA)x = B(Ax) = BT_A(x) = T_B\big(T_A(x)\big) = \big(T_B \circ T_A\big)(x)$.

Similarly, $T_{AB}(x) = (AB)x = A(Bx) = A\big(T_B(x)\big) = T_A\big(T_B(x)\big) = \big(T_A \circ T_B\big)(x)$.

Problem 9: Let A be a 2×2 matrix. Show that the image of the unit square $[0,1] \times [0,1]$ is a parallelogram with area $|\det A|$. Then show that the image of any square in the plane is a parallelogram with area equal to $\det A$ times the area of the original square.

Solution: Let $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

The unit square $[0,1] \times [0,1]$ is given by $\{ue_1 + ve_2 : 0 \le u \le 1, 0 \le v \le 1\}$.

The image of the unit square under A is then given by $\{uAe_1 + vAe_2 : 0 \le u, v \le 1\}$.

This is the parallelogram whose adjacent sides are $Ae_1 = \begin{bmatrix} a \\ c \end{bmatrix}$ and $Ae_2 = \begin{bmatrix} b \\ d \end{bmatrix}$.

The area of this parallelogram is $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = |ad - bc| = |\det A|$.

Now suppose we have any square (not necessarily the unit square). Since rotation does not change area we can assume the square to be $[0,l] \times [0,l]$ where l is the length of its side. The sides of this square are given by vectors le_1 and le_2. By linearity of A the sides of the parallelogram which is the image of this square under A are given by vectors lAe_1, lAe_2.
The area of this parallelogram is determinant of the matrix with column vectors lAe_1, lAe_2, that is, $\det(lA) = l^2 \det A = \det A$ times the area of the original square.

Problem 10: Let A be a 3×3 matrix. Show that the image of the unit cube $[0,1]^3$ is a parallelepiped with volume $|\det A|$. Then show the image of any cube is a parallelepiped with volume equal to $|\det A|$ times the volume of the original cube. Use this to show that if A and B are two 3×3 matrices then $\det(AB) = \det A \det B$.

Solution: Let $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

The unit cube $[0,1]^3$ is given by $\{ue_1 + ve_2 + te_3 : 0 \le u \le 1, 0 \le v \le 1, 0 \le t \le 1\}$.

The image of the unit cube under A is then given by $\{uAe_1 + vAe_2 + tAe_3 : 0 \le u, v, t \le 1\}$.

This is the parallelopiped whose adjacent sides are Ae_1, Ae_2 and Ae_3 which are respectively the first, second and third columns of A. The volume of such a parallelopiped is $|\det A|$.

Now suppose we have any cube (not necessarily the unit cube). Since rotation does not change volume we can assume the cube to be $[0,l]^3$ where l is the length of its side. The sides of this cube are given by vectors le_1, le_2, le_3. By linearity of A the sides of the parallelopiped which is the image of this cube under A are given by vectors lAe_1, lAe_2, lAe_3.
The volume of this parallelopiped is determinant of the matrix with column vectors lAe_1, lAe_2, lAe_3, which is, $|\det(lA)| = l^3|\det A| = |\det A|$ times the volume of the original cube.

The oriented volume of the parallelopiped, say P, which is the image under AB of the unit cube is $\det(AB)$. This same volume can also be computed by first applying B to the unit cube which is a parallolopiped with oriented volume $\det B$, and then applying A to this parallelopiped to get the parallelopiped P which will have oriented volume $\det A$ times $\det B$.

10. Eigenvalues and Eigenvectors

Problem 1: Find the eigenvalues and eigenvectors for the matrices

$$\begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix} \quad \begin{bmatrix} 5 & 1 \\ 0 & -2 \end{bmatrix} \quad \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$$

Solution: (1)

$$A = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$$

Then $c_A(\lambda) = \begin{vmatrix} 5-\lambda & 0 \\ 0 & -2-\lambda \end{vmatrix} = (5-\lambda)(-2-\lambda)$.

$c_A(\lambda) = 0 \Rightarrow (5-\lambda)(-2-\lambda) = 0 \Rightarrow \lambda = 5$ or -2. Thus, the $\boxed{\text{eigenvalues of } A \text{ are } 5 \text{ and } -2}$.

Notice that these are the diagonal entries of the given diagonal matrix.

Now let $\begin{bmatrix} x \\ y \end{bmatrix}$ be the eigenvector corresponding to $\lambda = 5$. Then $\begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 5 \begin{bmatrix} x \\ y \end{bmatrix}$

So we have $5x = 5x$ and $-2y = 5y \Rightarrow 7y = 0 \Rightarrow y = 0$ and x can be any real number.

Thus, the eigenvectors corresponding to $\lambda = 5$ are $\begin{bmatrix} x \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ where $x \in \mathbb{R}$.

In particular, $\boxed{\text{an eigenvector corresponding to the eigenvalue of 5 is } \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$.

Now let $\begin{bmatrix} x \\ y \end{bmatrix}$ be the eigenvector corresponding to $\lambda = -2$. Then $\begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -2 \begin{bmatrix} x \\ y \end{bmatrix}$

So we have $5x = -2x$ and $-2y = -2y \Rightarrow 7x = 0 \Rightarrow x = 0$ and y can be any real number.

Thus, the eigenvectors corresponding to $\lambda = -2$ are $\begin{bmatrix} 0 \\ y \end{bmatrix} = y \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ where $y \in \mathbb{R}$.

In particular, $\boxed{\text{an eigenvector corresponding to the eigenvalue of } -2 \text{ is } \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$.

(2)

$$A = \begin{bmatrix} 5 & 1 \\ 0 & -2 \end{bmatrix}$$

Then $c_A(\lambda) = \begin{vmatrix} 5-\lambda & 1 \\ 0 & -2-\lambda \end{vmatrix} = (5-\lambda)(-2-\lambda)$.

$c_A(\lambda) = 0 \Rightarrow (5-\lambda)(-2-\lambda) = 0 \Rightarrow \lambda = 5$ or -2. Thus, the $\boxed{\text{eigenvalues of } A \text{ are } 5 \text{ and } -2}$.

Notice that these are the diagonal entries of the given upper triangular matrix.

Now let $\begin{bmatrix} x \\ y \end{bmatrix}$ be the eigenvector corresponding to $\lambda = 5$. Then $\begin{bmatrix} 5 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 5 \begin{bmatrix} x \\ y \end{bmatrix}$

So we have $5x + y = 5x$ and $-2y = 5y \Rightarrow \Rightarrow y = 0$ and x can be any real number.

Thus, the eigenvectors corresponding to $\lambda = 5$ are $\begin{bmatrix} x \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ where $x \in \mathbb{R}$.

In particular, $\boxed{\text{an eigenvector corresponding to the eigenvalue of 5 is } \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$.

Now let $\begin{bmatrix} x \\ y \end{bmatrix}$ be the eigenvector corresponding to $\lambda = -2$. Then $\begin{bmatrix} 5 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -2 \begin{bmatrix} x \\ y \end{bmatrix}$

So we have $5x + y = -2x$ and $-2y = -2y \Rightarrow y = -7x$ and x can be any real number.

Thus, the eigenvectors corresponding to $\lambda = -2$ are $\begin{bmatrix} x \\ -7x \end{bmatrix} = x \begin{bmatrix} 1 \\ -7 \end{bmatrix}$ where $x \in \mathbb{R}$.

In particular, $\boxed{\text{an eigenvector corresponding to the eigenvalue of } -2 \text{ is } \begin{bmatrix} 1 \\ -7 \end{bmatrix}}$.

(3)

$$A = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$$

Then $c_A(\lambda) = \begin{vmatrix} 5-\lambda & 1 \\ 0 & 5-\lambda \end{vmatrix} = (5-\lambda)^2$.

$c_A(\lambda) = 0 \Rightarrow (5-\lambda)^2 = 0 \Rightarrow \lambda = 5$. Thus, the $\boxed{\text{only eigenvalue of } A \text{ is } 5}$.

Notice that the diagonal entries of the given upper triangular matrix are also 5.

Now let $\begin{bmatrix} x \\ y \end{bmatrix}$ be the eigenvector corresponding to $\lambda = 5$. Then $\begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 5 \begin{bmatrix} x \\ y \end{bmatrix}$

So we have $5x + y = 5x$ and $5y = 5y \Rightarrow y = 0$ and x can be any real number.

Thus, the eigenvectors corresponding to $\lambda = 5$ are $\begin{bmatrix} x \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ where $x \in \mathbb{R}$.

In particular, $\boxed{\text{an eigenvector corresponding to the eigenvalue of 5 is } \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$.

Problem 2: Find the eigenvalues and eigenvectors for the matrices

$$\begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix} \quad \begin{bmatrix} -1 & 3 \\ -3 & 5 \end{bmatrix} \quad \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix}$$

Solution: (1)

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix}$$

Then $c_A(\lambda) = \begin{vmatrix} 1-\lambda & 3 \\ 2 & -4-\lambda \end{vmatrix} = (1-\lambda)(-4-\lambda) - 6 = \lambda^2 + 3\lambda - 10 = (\lambda+5)(\lambda-2)$.

$c_A(\lambda) = 0 \Rightarrow (\lambda+5)(\lambda-2) = 0 \Rightarrow \lambda = -5$ or 2. Thus, the $\boxed{\text{eigenvalues of } A \text{ are } -5 \text{ and } 2}$.

Now let $\begin{bmatrix} x \\ y \end{bmatrix}$ be the eigenvector corresponding to $\lambda = -5$. Then $\begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -5 \begin{bmatrix} x \\ y \end{bmatrix}$

So we have $x + 3y = -5x$ and $2x - 4y = -5y \Rightarrow y = -2x$ and x can be any real number.

Thus, the eigenvectors corresponding to $\lambda = -5$ are $\begin{bmatrix} x \\ -2x \end{bmatrix} = x \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ where $x \in \mathbb{R}$.

In particular, $\boxed{\text{an eigenvector corresponding to the eigenvalue of } -5 \text{ is } \begin{bmatrix} 1 \\ -2 \end{bmatrix}}$.

Now let $\begin{bmatrix} x \\ y \end{bmatrix}$ be the eigenvector corresponding to $\lambda = 2$. Then $\begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$

So we have $x + 3y = 2x$ and $2x - 4y = 2y \Rightarrow x = 3y$ and y can be any real number.

Thus, the eigenvectors corresponding to $\lambda = 2$ are $\begin{bmatrix} 3y \\ y \end{bmatrix} = y \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ where $y \in \mathbb{R}$.

In particular, $\boxed{\text{an eigenvector corresponding to the eigenvalue of 2 is } \begin{bmatrix} 3 \\ 1 \end{bmatrix}}$.

(2)

$$A = \begin{bmatrix} -1 & 3 \\ -3 & 5 \end{bmatrix}$$

Then $c_A(\lambda) = \begin{vmatrix} -1-\lambda & 3 \\ -3 & 5-\lambda \end{vmatrix} = (-1-\lambda)(5-\lambda) + 9 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$.

$c_A(\lambda) = 0 \Rightarrow (\lambda - 2)^2 = 0 \Rightarrow \lambda = 2$. Thus, the $\boxed{\text{only eigenvalue of } A \text{ is } 2}$.

Now let $\begin{bmatrix} x \\ y \end{bmatrix}$ be the eigenvector corresponding to $\lambda = 2$. Then $\begin{bmatrix} -1 & 3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$

So we have $-x + 3y = 2x$ and $-3x + 5y = 2y \Rightarrow y = x$ and x can be any real number.

Thus, the eigenvectors corresponding to $\lambda = 2$ are $\begin{bmatrix} x \\ x \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ where $x \in \mathbb{R}$.

In particular, $\boxed{\text{an eigenvector corresponding to the eigenvalue of 2 is } \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$.

(3)

$$A = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix}$$

Then $c_A(\lambda) = \begin{vmatrix} 1-\lambda & -5 \\ 1 & -3-\lambda \end{vmatrix} = (1-\lambda)(-3-\lambda) + 5 = \lambda^2 + 2\lambda + 2$.

$c_A(\lambda) = 0$ has no real roots. Thus, the given matrix has $\boxed{\text{no real eigenvalues}}$.

Problem 3: Find the eigenvalues and eigenvectors for the matrices

$$\begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} \quad \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} \quad \begin{bmatrix} -4 & 5 \\ -1 & -2 \end{bmatrix} \quad \begin{bmatrix} -1 & 5 \\ -1 & 3 \end{bmatrix}$$

Solution: (1)

$$A = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix}$$

Then $c_A(\lambda) = \begin{vmatrix} -\lambda & 1 \\ 3 & -2-\lambda \end{vmatrix} = (-\lambda)(-2-\lambda) - 3 = \lambda^2 + 2\lambda - 3 = (\lambda+3)(\lambda-1)$.

$c_A(\lambda) = 0 \Rightarrow (\lambda+3)(\lambda-1) = 0 \Rightarrow \lambda = -3$ or 1. Thus, the $\boxed{\text{eigenvalues of } A \text{ are } -3 \text{ and } 1}$.

Now let $\begin{bmatrix} x \\ y \end{bmatrix}$ be the eigenvector corresponding to $\lambda = -3$. Then $\begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -3 \begin{bmatrix} x \\ y \end{bmatrix}$

So we have $y = -3x$ and $3x - 2y = -3y \Rightarrow y = -3x$ and x can be any real number.

Thus, the eigenvectors corresponding to $\lambda = -3$ are $\begin{bmatrix} x \\ -3x \end{bmatrix} = x \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ where $x \in \mathbb{R}$.

In particular, $\boxed{\text{an eigenvector corresponding to the eigenvalue of } -3 \text{ is } \begin{bmatrix} 1 \\ -3 \end{bmatrix}}$.

Now let $\begin{bmatrix} x \\ y \end{bmatrix}$ be the eigenvector corresponding to $\lambda = 1$. Then $\begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

So we have $y = x$ and $3x - 2y = y \Rightarrow y = x$ and x can be any real number.

Thus, the eigenvectors corresponding to $\lambda = 1$ are $\begin{bmatrix} x \\ x \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ where $x \in \mathbb{R}$.

In particular, $\boxed{\text{an eigenvector corresponding to the eigenvalue of 1 is } \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$.

(2)

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$$

Then $c_A(\lambda) = \begin{vmatrix} 2-\lambda & 1 \\ 3 & -\lambda \end{vmatrix} = (2-\lambda)(-\lambda) - 3 = \lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1)$.

$c_A(\lambda) = 0 \Rightarrow (\lambda-3)(\lambda+1) = 0 \Rightarrow \lambda = 3$ or -1. Thus, the $\boxed{\text{eigenvalues of } A \text{ are 3 and } -1}$.

Now let $\begin{bmatrix} x \\ y \end{bmatrix}$ be the eigenvector corresponding to $\lambda = 3$. Then $\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix}$

So we have $2x + y = 3x$ and $3x = 3y \Rightarrow y = x$ and x can be any real number.

Thus, the eigenvectors corresponding to $\lambda = 3$ are $\begin{bmatrix} x \\ x \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ where $x \in \mathbb{R}$.

In particular, $\boxed{\text{an eigenvector corresponding to the eigenvalue of 3 is } \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$.

Now let $\begin{bmatrix} x \\ y \end{bmatrix}$ be the eigenvector corresponding to $\lambda = -1$. Then $\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -1 \begin{bmatrix} x \\ y \end{bmatrix}$

So we have $2x + y = -x$ and $3x = -y \Rightarrow y = -3x$ and x can be any real number.

Thus, the eigenvectors corresponding to $\lambda = -1$ are $\begin{bmatrix} x \\ -3x \end{bmatrix} = x \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ where $x \in \mathbb{R}$.

In particular, $\boxed{\text{an eigenvector corresponding to the eigenvalue of } -1 \text{ is } \begin{bmatrix} 1 \\ -3 \end{bmatrix}}$.

(3)

$$A = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$$

Then $c_A(\lambda) = \begin{vmatrix} 3-\lambda & 2 \\ -1 & 1-\lambda \end{vmatrix} = (3-\lambda)(1-\lambda) + 2 = \lambda^2 - 4\lambda + 5.$

$c_A(\lambda) = 0$ has no real roots. Thus, the given matrix has $\boxed{\text{no real eigenvalues}}$.

(4)

$$A = \begin{bmatrix} -4 & 5 \\ -1 & -2 \end{bmatrix}$$

Then $c_A(\lambda) = \begin{vmatrix} -4-\lambda & 5 \\ -1 & -2-\lambda \end{vmatrix} = (-4-\lambda)(-2-\lambda) + 5 = \lambda^2 + 6\lambda + 13.$

$c_A(\lambda) = 0$ has no real roots. Thus, the given matrix has $\boxed{\text{no real eigenvalues}}$.

(5)

$$A = \begin{bmatrix} -1 & 5 \\ -1 & 3 \end{bmatrix}$$

Then $c_A(\lambda) = \begin{vmatrix} -1-\lambda & 5 \\ -1 & 3-\lambda \end{vmatrix} = (-1-\lambda)(3-\lambda) + 5 = \lambda^2 - 2\lambda + 2.$

$c_A(\lambda) = 0$ has no real roots. Thus, the given matrix has $\boxed{\text{no real eigenvalues}}$.

Problem 4: Find the eigenvalues and eigenvectors for the matrices

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 0 \\ 2 & 1 & -6 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \quad \begin{bmatrix} -1 & -1 & 1 \\ 1 & -2 & 3 \\ 1 & 3 & -2 \end{bmatrix}$$

Solution: (1)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 0 \\ 2 & 1 & -6 \end{bmatrix}$$

Then $c_A(\lambda) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -2-\lambda & 0 \\ 2 & 1 & -6-\lambda \end{vmatrix} = \underbrace{(-6-\lambda)\big(-\lambda(-2-\lambda)\big)}_{\text{Expanding by first column}} = -\lambda(\lambda+2)(\lambda+6)$. The real roots of $c_A(\lambda) = 0$ are $\lambda = 0, -2, -6$. Thus, the $\boxed{\text{eigenvalues of } A \text{ are } \lambda = 0, -2, -6}$.

(2)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

Then $c_A(\lambda) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & -\lambda & 2 \\ 0 & 1 & -1-\lambda \end{vmatrix} = \underbrace{(1-\lambda)(\lambda^2+\lambda-2)}_{\text{(Expanding by first column)}} = -(\lambda-1)^2(\lambda+2)$

The real roots of $c_A(\lambda) = 0$ are $\lambda = 1, -2$. Thus, the $\boxed{\text{eigenvalues of } A \text{ are } \lambda = 1 \text{ and } \lambda = -2}$.

(3)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

Then $c_A(\lambda) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & -1 & 1-\lambda \end{vmatrix} = \underbrace{-\lambda(\lambda^2-\lambda+1)+1(1)}_{\text{(Expanding by first column)}} = -(\lambda^3-\lambda^2+\lambda-1)$

$= -(\lambda-1)(\lambda^2+1)$

The real roots of $c_A(\lambda) = 0$ are $\lambda = 1$. Thus, the only (real) $\boxed{\text{eigenvalue of } A \text{ is } \lambda = 1}$.

(4)

$$A = \begin{bmatrix} -1 & -1 & 1 \\ 1 & -2 & 3 \\ 1 & 3 & -2 \end{bmatrix}$$

Then

$$c_A(\lambda) = \begin{vmatrix} -1-\lambda & -1 & 1 \\ 1 & -2-\lambda & 3 \\ 1 & 3 & -2-\lambda \end{vmatrix} = \underbrace{(-1-\lambda)\begin{vmatrix} -2-\lambda & 3 \\ 3 & -2-\lambda \end{vmatrix} - 1\begin{vmatrix} -1 & 1 \\ 3 & -2-\lambda \end{vmatrix} + 1\begin{vmatrix} -1 & 1 \\ -2-\lambda & 3 \end{vmatrix}}_{\text{Expanding by first column}}$$

$= (-1-\lambda)(\lambda^2+4\lambda-5) - (\lambda-1) + (\lambda-1 = -(\lambda+1)(\lambda^2+4\lambda-5) = -(\lambda+1)(\lambda-1)(\lambda+5)$

The real roots of $c_A(\lambda) = 0$ are $\lambda = 1, -1, -5$. Thus, the $\boxed{\text{eigenvalues of } A \text{ are } \lambda = 1, -1, -5}$.

Problem 5: Find the eigenvalues for the matrices

$$\begin{bmatrix} -1 & 2 & 0 \\ 3 & 0 & -2 \\ 0 & 3 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & 2 \\ 1 & -3 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 1 \\ -2 & 2 & -2 \\ 0 & -3 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

Solution: (1)

$$A = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 0 & -2 \\ 0 & 3 & -1 \end{bmatrix}$$

Then $c_A(\lambda) = \begin{vmatrix} -1-\lambda & 2 & 0 \\ 3 & -\lambda & -2 \\ 0 & 3 & -1-\lambda \end{vmatrix} = \underbrace{(-1-\lambda)(\lambda^2+\lambda+6) - 3(-2-\lambda)}_{\text{(Expanding by first column)}}$

$$= -\lambda^3 - 2\lambda^2 - 7\lambda - 6 + 6 + 3\lambda = -\lambda(\lambda^2+2\lambda+4)$$

The real roots of $c_A(\lambda) = 0$ are $\lambda = 0$. Thus, the (real) $\boxed{\text{eigenvalues of } A \text{ are } \lambda = 0}$.

(2)

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & 2 \\ 1 & -3 & 0 \end{bmatrix}$$

Then

$$c_A(\lambda) = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 1 & -2-\lambda & 2 \\ 1 & -3 & -\lambda \end{vmatrix} = \underbrace{(1-\lambda)(\lambda^2+2\lambda+6)+1(-3+2+\lambda)}_{\text{(Expanding along first row)}} = -(\lambda-1)(\lambda^2+2\lambda+5)$$

The real roots of $c_A(\lambda) = 0$ are $\lambda = 1$. Thus, the (real) $\boxed{\text{eigenvalues of } A \text{ are } \lambda = 1}$.

(3)

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 2 & -2 \\ 0 & -3 & 0 \end{bmatrix}$$

Then $c_A(\lambda) = \begin{vmatrix} 1-\lambda & 2 & 1 \\ -2 & 2-\lambda & -2 \\ 0 & -3 & -\lambda \end{vmatrix} = \underbrace{(1-\lambda)(\lambda^2-2\lambda-6)+2(-2\lambda+3)}_{\text{(Expanding along first column)}}$

$$= -\lambda^3+3\lambda^2+4\lambda-6-4\lambda+6 = -\lambda^2(\lambda-3)$$

The real roots of $c_A(\lambda) = 0$ are $\lambda = 0, 3$. Thus, the $\boxed{\text{eigenvalues of } A \text{ are } \lambda = 0 \text{ and } \lambda = 3}$.

(4)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

Then $c_A(\lambda) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = \underbrace{-\lambda(2-3\lambda+\lambda^2+1)+1(1)}_{\text{Expanding along first column}}$

$$= -\lambda^3+3\lambda^2-3\lambda+1 = -(\lambda-1)^3$$

The real roots of $c_A(\lambda) = 0$ are $\lambda = 1$. Thus, the $\boxed{\text{only eigenvalue of } A \text{ is } \lambda = 1}$.

Problem 6: Show geometrically by looking at the image of the function defined by a 2×2 matrix that a 2×2 matrix is not invertible if and only if it has 0 for an eigenvalue.

Solution: If a matrix A is not invertible, then there is some line passing through origin which is mapped by A to origin. Take vector $\vec{x}$ to be the direction vector of this line and we have $A\vec{x} = 0 = 0.\vec{x}$ for $\vec{x} \neq 0$, and hence A has 0 as an eigenvalue.

If a matrix A has 0 as an eigenvalue then there is a nonzero vector $\vec{x}$ such that $A\vec{x} = 0.\vec{x} = 0$, which means that A maps the line passing through origin in the direction of the vector $\vec{x}$ to origin. Thus, A is not invertible.

Problem 7: Show that for each $c \in \mathbb{R}$ the function defined by the matrix $A = \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}$ maps the graphs of the hyperbola $xy = c$ to itself.

Solution: Any point on the graph of $xy = c$ is of the form $(t, c/t)$ where $t \neq 0$.

$$\begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} t \\ c/t \end{bmatrix} = \begin{bmatrix} t/2 \\ 2c/t \end{bmatrix}$$

The point $(t/2, 2c/t)$ obtained after applying A satisfies $xy = (t/2)(2c/t) = c$.

Problem 8: Show that the function defined by the matrix $D = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$ maps every circle centered at the origin to itself. Then prove it rotates the circle by $30°$.

Solution: Any point on the circle $x^2 + y^2 = r^2$ is given by $(r\cos\theta, r\sin\theta)$.

$$\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} = \begin{bmatrix} r\frac{\sqrt{3}}{2}\cos\theta - r\frac{1}{2}\sin\theta \\ r\frac{1}{2}\cos\theta + r\frac{\sqrt{3}}{2}\sin\theta \end{bmatrix} = \begin{bmatrix} r\cos(\theta + \pi/6) \\ r\sin(\theta + \pi/6) \end{bmatrix}$$

Then for the point obtained after applying A, we have

$$x^2 + y^2 = \big(r\cos(\theta + \pi/6)\big)^2 + \big(r\sin(\theta + \pi/6)\big)^2 = r^2.$$

which is again the same circle.

We also see that the output point obtained after applying A is same as point obtained after rotating the input point by $\pi/6 = 30°$.