Parametric Curves

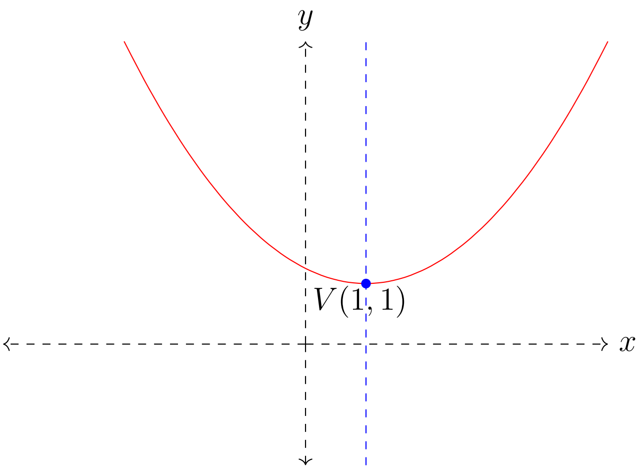
Problem 1: Sketch the following parametric curves.

1. $x = 2t + 1$, $y = t^2 + 1$, $t \in \mathbb{R}$.
2. $x = 1 + \sin\theta$, $y = -1 + 2\cos\theta$, $0 \leq \theta \leq 2\pi$.
3. $x = 2 + 2\sec\theta$, $y = 1 + 4\tan\theta$, $\theta \in (-\pi/2, \pi/2) \bigcup (\pi/2, 3\pi/2)$.
4. $x = t$, $y = 4 - t$, $0 \leq t \leq 4$.

Solutions. (1) Eliminating the parameter t, we have

$$y - 1 = t^2 = \left(\frac{x-1}{2}\right)^2 \quad \Rightarrow (y-1) = \frac{1}{4}(x-1)^2 \text{ or } (x-1)^2 = 4(y-1)$$

Shifting the origin at $(1, 1)$ we get $X^2 = 4Y$, which the equation of a parabola.



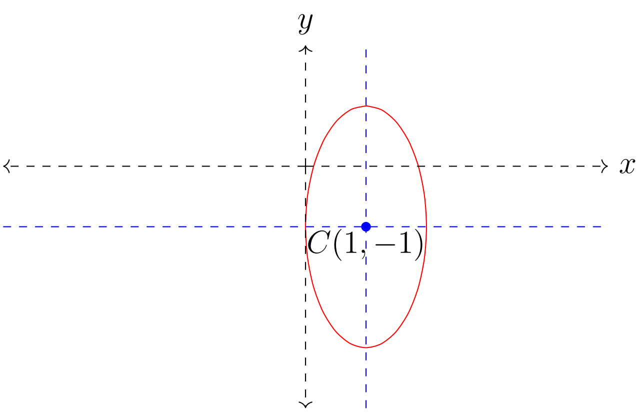
(2)

$$x = 1 + \sin\theta \Rightarrow \sin\theta = (x-1) \text{ and } y = -1 + 2\cos\theta \Rightarrow \cos\theta = \frac{y+1}{2}$$

Since $\sin^2\theta + \cos^2\theta = 1$ we have

$$(x-1)^2 + \frac{(y+1)^2}{4} = 1$$

which is the equation of an ellipse in standard form 2 (major axis parallel to y-axis) with center at $(1, -1)$, $a = 1$ and $b = 2$.



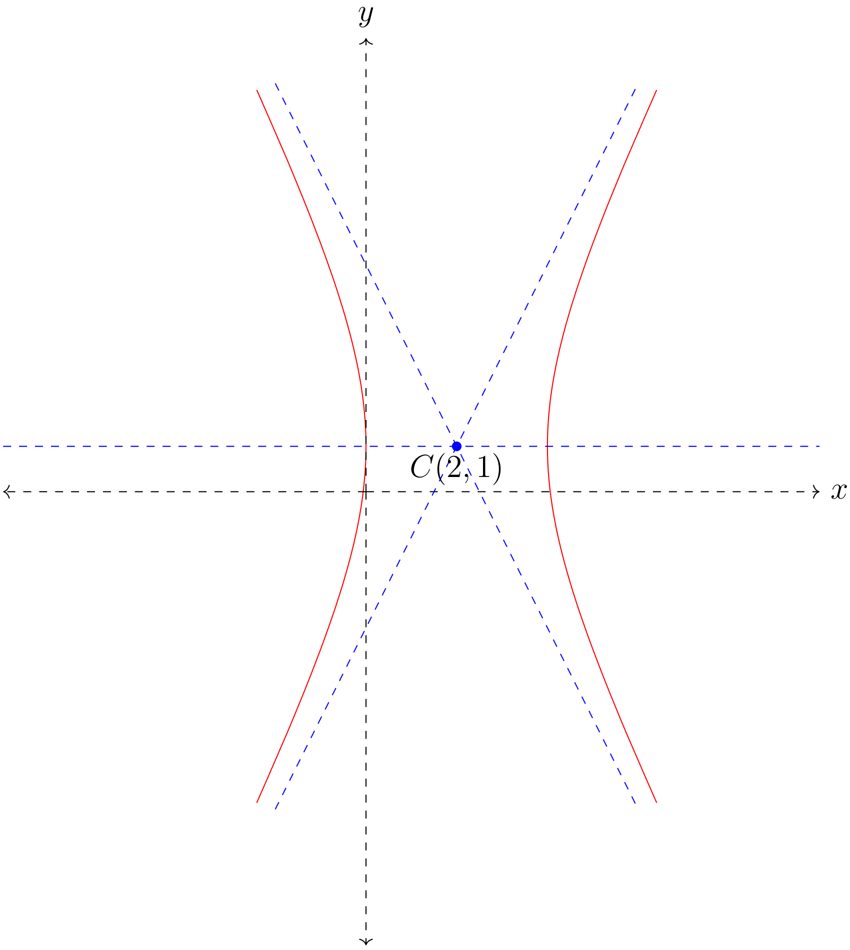
(3)

$$x = 2 + 2\sec\theta \Rightarrow \sec\theta = \frac{x-2}{2} \text{ and } y = 1 + 4\tan\theta \Rightarrow \tan\theta = \frac{y-1}{4}$$

Since $\sec^2\theta - \tan^2\theta = 1$ we have

$$\frac{(x-2)^2}{4} - \frac{(y-1)^2}{16} = 1$$

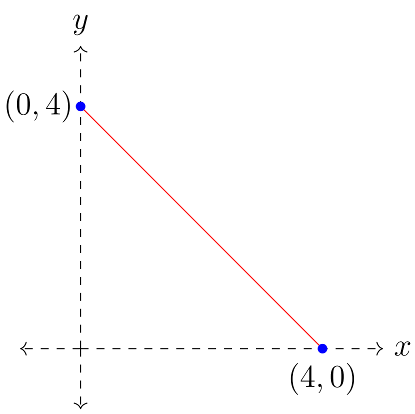
which is the equation of a hyperbola in standard form 1 (axis parallel to x-axis) with center at $(2,1)$, $a = 2$ and $b = 4$.



(4)

$$x = t \text{ and } y = 4 - t \Rightarrow y = 4 - x \text{ or } x + y = 4$$

which is cartesian equation of a straight line. But since the parameter t varies between $0 \leq t \leq 4$, the given parametric equation represents a line segment as shown in the graph:-



□

Problem 2: Eliminate the parameter to find the cartesian equation for the following parametric curves.

1. $x = \sqrt{t}$, $y = 1 - t$.
2. $x = t^2$, $y = \ln t$.
3. $x = t^2$, $y = t^3$.

Solutions. (1)

$$x = \sqrt{t} \Rightarrow t = x^2$$

Substituting the value of t in $y = 1 - t$ we get $y = 1 - x^2$ or $\boxed{x^2 + y = 1}$ which is the required cartesian equation.

(2)

$$y = \ln t \Rightarrow t = e^y$$

Substituting the value of t in $x = t^2$ we get:-

$$\boxed{x = e^{2y}.}$$

(3)

$$x = t^2 \Rightarrow x^3 = t^6 \text{ and } y = t^3 \Rightarrow y^2 = t^6$$

Therefore,

$$\boxed{x^3 = y^2}$$

is the required cartesian equation. □

Problem 3: Find the parametric equation of the following conic sections.

1. A parabola with vertex at $(2, 2)$ and focus at $(3, 2)$.
2. An ellipse with center at $(-1, 4)$, a vertex at $(-1, 0)$ and a focus at $(-1, 6)$
3. A hyperbola with foci at $(2, 0)$, $(2, 8)$ and asymptotes $y = 3 + \frac{1}{2}x$, $y = 5 - \frac{1}{2}x$.

Solutions. (1) Shift the origin to the vertex $(2, 2)$ to obtain new coordinates

$$X = x - 2 \text{ and } Y = y - 2$$

In the new coordinates, the focus is at $(3 - 2, 2 - 2) = (1, 0)$. Thus, $p = 1$.
Since the focus lie on +ve X-axis, the parametric equation is given by

$$X = 1t^2,\ Y = 2(1)t \text{ or } x - 2 = t^2,\ y - 2 = 2t$$

that is

$$\boxed{x = 2 + t^2,\ y = 2 + 2t \text{ where } t \in \mathbb{R}.}$$

(2) Shift the origin at the center $(-1, 4)$ to obtain new coordinates

$$X = x + 1 \text{ and } Y = y - 4$$

In the new coordinates, a vertex is at $(-1+1, 0-4) = (0, -4)$
and a focus is at $(-1+1, 6-4) = (0, 2)$.

Therefore, $a = 4$ and $c = 2$. This implies,

$$b^2 = a^2 - c^2 = 4^2 - 2^2 = 16 - 4 = 12 \Rightarrow b = \sqrt{12} = 2\sqrt{3}$$

Since focus and vertex lie on Y-axis, the parametric equation of this ellipse is given by:-

$$X = b\cos\theta,\ Y = a\sin\theta \Rightarrow x + 1 = 2\sqrt{3}\cos\theta,\ y - 4 = 4\sin\theta$$

that is

$$\boxed{x = -1 + 2\sqrt{3}\cos\theta,\ y = 4 + 4\sin\theta \text{ where } \theta \in [0, 2\pi).}$$

(3) We first compute intersection of the asymptotes $y = 3 + \frac{1}{2}x$, $y = 5 - \frac{1}{2}x$ to find center.

$$3 + \frac{1}{2}x = 5 - \frac{1}{2}x \Rightarrow x = 5 - 3 = 2 \Rightarrow y = 3 + \frac{1}{2} \times 2 = 4$$

Therefore, center of the hyperbola lies at $(2, 4)$. Shift the origin at $(2, 4)$ to obtain new coordinates

$$X = x - 2 \text{ and } Y = y - 4$$

The foci in new coordinates are at

$$(2-2, 0-4) = (0, -4) \text{ and } (2-2, 8-4) = (0, 4)$$

Therefore,

$$c = 4 \Rightarrow c^2 = a^2 + b^2 = 16 \ldots\ldots\ldots\ldots (*)$$

Since the foci lie on Y-axis, the slope of asymptotes is given by $\pm\dfrac{a}{b} = \pm\dfrac{1}{2}$. Thus, $b = 2a$.
Substituting $b = 2a$ in $(*)$ we have

$$a^2 + (2a)^2 = 16 \Rightarrow a^2 = \frac{16}{5} \Rightarrow a = \frac{4}{\sqrt{5}} \text{ and } b = \frac{8}{\sqrt{5}}$$

Since the foci are on Y-axis, the parametric equation of the hyperbola is given by:-

$$Y = a\sec\theta,\ X = b\tan\theta \Rightarrow y - 4 = \frac{4}{\sqrt{5}}\sec\theta,\ x - 2 = \frac{8}{\sqrt{5}}\tan\theta$$

Thus, the parametric equation of the given hyperbola is:-

$$\boxed{x = 2 + \frac{8}{\sqrt{5}}\tan\theta,\ y = 4 + \frac{4}{\sqrt{5}}\sec\theta \text{ where } \theta \in [0, 2\pi) \setminus \{\pi/2, 3\pi/2\}.}$$

$\square$

Polar Coordinates

Problem 1: Find the Cartesian coordinates of points whose polar coordinates are as follows:-

$$(3, -\pi/3) \quad , \quad (-2, 3\pi/2) \quad , \quad (-1, 5\pi/4)$$

Solution.

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

Therefore,

$$(3, -\pi/3) \equiv \big(3\cos(-\pi/3), 3\sin(-\pi/3)\big) = \left(\frac{3}{2}, -\frac{3\sqrt{3}}{2}\right)$$
$$(-2, 3\pi/2) \equiv \big(-2\cos(3\pi/2), -2\sin(3\pi/2)\big) = (0, 2)$$
$$(-1, 5\pi/4) \equiv \big(-1\cos(5\pi/4), -1\sin(5\pi/4)\big) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

□

Problem 2: Find the polar coordinates of points whose Cartesian coordinates are as follows:-

$$(-4, 4) \quad , \quad (\sqrt{3}, -1) \quad , \quad (-6, 0)$$

Solution.

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

For $(-4, 4)$,

$$r = \sqrt{(-4)^2 + (4)^2} = 4\sqrt{2} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{4}{-4}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

Therefore, $(-4, 4) \equiv (4\sqrt{2}, 3\pi/4)$.

For $(\sqrt{3}, -1)$,

$$r = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2 \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$$

Therefore, $(\sqrt{3}, -1) \equiv (2, 11\pi/6)$.

For $(-6, 0)$,

$$r = \sqrt{(-6)^2 + (0)^2} = 6 \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{0}{-6}\right) = \pi - 0 = \pi$$

Therefore, $(-6, 0) \equiv (6, \pi)$. □

Problem 3: Identify the curves by finding their Cartesian equations.

1. $r = 4\sec\theta$
2. $r = 5\cos\theta$
3. $r^2\cos 2\theta = 1$

Solution.

$$r = \sqrt{x^2+y^2} \quad , \quad \cos\theta = \frac{x}{\sqrt{x^2+y^2}} \quad , \quad \sin\theta = \frac{y}{\sqrt{x^2+y^2}}$$

(1)

$$r = 4\sec\theta \Rightarrow r\cos\theta = 4 \Rightarrow \sqrt{x^2+y^2}\frac{x}{\sqrt{x^2+y^2}} = 4 \Rightarrow x = 4$$

Therefore, the given equation represents a vertical straight line passing through $(4, 0)$.

(2)

$$r = 5\cos\theta \Rightarrow \sqrt{x^2+y^2} = 5\frac{x}{\sqrt{x^2+y^2}} \Rightarrow (x^2+y^2) = 5x$$

$$\Rightarrow x^2 - 5x + y^2 = 0 \Rightarrow \underbrace{x^2 - 2\left(\frac{5}{2}\right)x + \left(\frac{5}{2}\right)^2}\, - \left(\frac{5}{2}\right)^2 + y^2 = 0$$

$$\Rightarrow \left(x - \frac{5}{2}\right)^2 + y^2 = \left(\frac{5}{2}\right)^2$$

Therefore, the given equation represents a circle with centre at $(2.5, 0)$ and radius 2.5.

(3)

$$r^2\cos 2\theta = 1 \Rightarrow r^2(\cos^2\theta - \sin^2\theta) = 1 \Rightarrow (r\cos\theta)^2 - (r\sin\theta)^2 = 1$$
$$\Rightarrow x^2 - y^2 = 1$$

Therefore, the given equation represents a hyperbola with center at $(0, 0)$, axis parallel to x-axis and with $a = b = 1$. □

Problem 4: Find a polar equation of the curve whose Cartesian equation is as follows:-

1. $4y^2 = x$ (a parabola)
2. $x^2 + 4y^2 - 2x = 3$ (an ellipse)
3. $x^2 + y^2 = 2x$ (a circle)

Solution.

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

(1)

$$4y^2 = x \Rightarrow 4(r\sin\theta)^2 = r\cos\theta \Rightarrow 4r^2\sin^2\theta = r\cos\theta$$

So either $r = 0$ or $4r\sin^2\theta = \cos\theta$. But for $\theta = \pi/2$, the point $(0, \pi/2)$ (representing the pole) satisfies $4r\sin^2\theta = \cos\theta$. Therefore, the polar equation of the given parabola is

$$\boxed{4r\sin^2\theta = \cos\theta}$$

(2)

$$x^2 + 4y^2 - 2x = 3 \Rightarrow (r\cos\theta)^2 + 4(r\sin\theta)^2 - 2r\cos\theta = 3$$
$$\Rightarrow r^2(\cos^2\theta + 4\sin^2\theta) - 2r\cos\theta = 3 \Rightarrow r^2(1 + 3\sin^2\theta) - 2r\cos\theta = 3$$

Therefore, the polar equation of the given ellipse is

$$\boxed{r^2(1 + 3\sin^2\theta) - 2r\cos\theta = 3}$$

(3)

$$x^2 + y^2 = 2x \Rightarrow (r\cos\theta)^2 + (r\sin\theta)^2 = 2r\cos\theta$$
$$\Rightarrow r^2(\sin^2\theta + \cos^2\theta) = 2r\cos\theta \Rightarrow r^2 = 2r\cos\theta$$

So either $r = 0$ or $r = 2\cos\theta$. But the point $(0, \pi/2)$ (representing the pole) satisfies the equation $r = 2\cos\theta$. Therefore, the polar equation of the given circle is

$$\boxed{r = 2\cos\theta}$$

□

Problem 5: Evaluate the following expressions and write your answers in the form $a + bi$.

1. $\dfrac{1+i}{1-i}$
2. $\overline{2i(1-i)}$
3. i^{103}
4. $\sqrt{-3}\sqrt{-12}$

Solution. (1)

$$\frac{1+i}{1-i} = \frac{1+i}{1-i} \times \frac{1+i}{1+i} = \frac{(1+i)(1+i)}{(1-i)(1+i)} = \frac{1+i+i+i^2}{1-i^2} = \frac{1+2i-1}{1-(-1)} = \frac{2i}{2} = i$$

(2)

$$\overline{2i(1-i)} = \overline{2i - 2i^2} = \overline{2i - 2(-1)} = \overline{2i+2} = \overline{2+2i} = 2 - 2i$$

(3)

$$i^4 = 1 \Rightarrow i^{103} = i^{100+3} = i^{4\times 25+3} = \left(i^4\right)^{25} i^3 = (1)^{25} i^3 = i^3 = i^2 \times i = (-1)i = -i$$

(4)

$$\sqrt{-3}\sqrt{-12} = \sqrt{3}i \times \sqrt{12}i = \sqrt{36}i^2 = 6i^2 = 6(-1) = -6$$

□

Complex Numbers

Problem 1: Let $z = 2\sqrt{3} - 2i$ and $w = -1 + i$. Find polar forms of zw, z/w and $1/z$ by putting z and w into polar forms.

Solution. The polar forms of z, w are given by

$$z = |z|(\cos\theta_1 + i\sin\theta_1) \quad , \quad w = |w|(\cos\theta_2 + i\sin\theta_2)$$

where $\theta_1 = \arg z$ and $\theta_2 = \arg w$.

For $z = 2\sqrt{3} - 2i$

$$|z| = \sqrt{(2\sqrt{3})^2 + (-2)^2} = 4 \quad \text{and} \quad \theta_1 = \tan^{-1}\left(\frac{-2}{2\sqrt{3}}\right) = -\frac{\pi}{6}$$

For $w = w = -1 + i$,

$$|w| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2} \quad \text{and} \quad \theta_2 = \tan^{-1}\left(\frac{1}{-1}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

Then

$$zw = |z||w|\big(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\big) \quad \text{and} \quad \frac{z}{w} = \frac{|z|}{|w|}\big(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\big)$$

$$\theta_1 + \theta_2 = -\frac{\pi}{6} + \frac{3\pi}{4} = \frac{7\pi}{12} \quad \text{and} \quad \theta_1 - \theta_2 = -\frac{\pi}{6} - \frac{3\pi}{4} = -\frac{11\pi}{12} = 2\pi - \frac{11\pi}{12} = \frac{13\pi}{12}$$

Therefore,

$$zw = 4\sqrt{2}\big(\cos(7\pi/12) + i\sin(7\pi/12)\big) \quad \text{and} \quad \frac{z}{w} = 2\sqrt{2}\big(\cos(13\pi/12) + i\sin(13\pi/12)\big)$$

$$\frac{1}{z} = \frac{1}{|z|}\big(\cos(-\theta_1) + i\sin(-\theta_1)\big) = \frac{1}{4}\big(\cos(\pi/6) + i\sin(\pi/6)\big)$$

□

Problem 2: Use De Moivre's Theorem to find a and b where $a + bi = (1 - \sqrt{3}i)^5$.

Solutions. Find the polar form of $z = 1 - \sqrt{3}i$ first.

$$|z| = \sqrt{(1)^2 + (\sqrt{3})^2} = 2 \quad \text{and} \quad \arg(z) = \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right) = -\frac{\pi}{3}$$

By De Moivre's theorem,

$$z^n = |z|^n\big(\cos(n\theta) + i\sin(n\theta)\big)$$

Therefore,

$$(1 - \sqrt{3}i)^5 = 2^5\big(\cos(5 \times \frac{-\pi}{3}) + i\sin(5 \times \frac{-\pi}{3})\big) = 32\big(\cos(-\frac{5\pi}{3}) + i\sin(-\frac{5\pi}{3})\big)$$

Now,

$$\cos\left(-\frac{5\pi}{3}\right) = \cos\left(\frac{5\pi}{3}\right) = \cos\left(\frac{6\pi - \pi}{3}\right) = \cos\left(2\pi - \frac{\pi}{3}\right) = \cos\left(-\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$\sin\left(-\frac{5\pi}{3}\right) = -\sin\left(\frac{5\pi}{3}\right) = -\sin\left(\frac{6\pi-\pi}{3}\right) = -\sin\left(2\pi - \frac{\pi}{3}\right) = -\sin\left(-\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

So, we have

$$a + bi = 32\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 16 + 16\sqrt{3}i$$

Hence, $a = 16$ and $b = 16\sqrt{3}$.

$\square$

Problem 3: Find all solutions of the equation $x^2 + 2x + 5 = 0$.

Solution. By the quadratic formula,

$$x = \frac{-2 \pm \sqrt{2^2 - 4.1.5}}{2.1} = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

So, the given equation has two solutions, namely, $-1 + 2i$ and $-1 - 2i$.

Alternatively, one can use completion of squares,

$$x^2 + 2x + 5 = 0 \Rightarrow \underbrace{x^2 + 2x + 1}_{(x+1)^2} + 4 = 0 \Rightarrow (x+1)^2 = -4 \Rightarrow x + 1 = \pm 2i \Rightarrow x = -1 \pm 2i$$

$\square$

Problem 4: Find all the cube roots of i and sketch them in the complex plane.

Solutions.

$$i = 1\big(\cos(\pi/2) + i\sin(\pi/2)\big)$$

So, we need to solve the equation

$$z^3 = 1\big(\cos(\pi/2) + i\sin(\pi/2)\big)$$

Let $z = r\big(\cos\theta + i\sin\theta\big)$. Then

$$r^3\big(\cos(3\theta) + i\sin(3\theta)\big) = 1\big(\cos(\pi/2) + i\sin(\pi/2)\big)$$

$$\Rightarrow r^3 = 1 \quad \text{and} \quad 3\theta = 2k\pi + \frac{\pi}{2} = (4k+1)\frac{\pi}{2}$$

$$\Rightarrow r = 1 \quad \text{and} \quad \theta = (4k+1)\frac{\pi}{6} \quad \text{for } k = 0, \pm 1, \pm 2, \pm 3, \cdots$$

But distinct values occur only for $k = 0, 1, 2$. So, we have

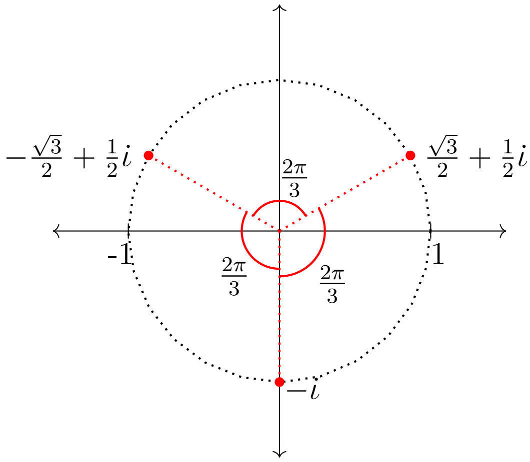
$$z = \cos(\pi/6) + i\sin(\pi/6) \quad \text{or} \quad z = \cos(5\pi/6) + i\sin(5\pi/6) \quad \text{or} \quad z = \cos(9\pi/6) + i\sin(9\pi/6)$$

$$\cos(5\pi/6) = -\cos(\pi/6) = -\sqrt{3}/2 \quad \text{and} \quad \sin(5\pi/6) = \sin(\pi/6) = 1/2$$

$$\cos(9\pi/6) = \cos(3\pi/2) = 0 \quad \text{and} \quad \sin(9\pi/6) = \sin(3\pi/2) = -1$$

Therefore, the three cube roots of i are

$$\frac{\sqrt{3}}{2} + \frac{1}{2}i \quad , \quad -\frac{\sqrt{3}}{2} + \frac{1}{2}i \quad , \quad -i$$



$\square$

Problem 5: Write the following numbers in the form $a + bi$.

$$e^{i\pi/3} \quad , \quad e^{-i\pi} \quad , \quad e^{2+i\pi}$$

Solutions. By Euler's Formula,

$$e^{i\theta} = \cos\theta + i\sin\theta$$

So, we have

$$e^{i\pi/3} = \cos(\pi/3) + i\sin(\pi/3) = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$e^{i\pi} = \cos(-\pi) + i\sin(-\pi) = -1 + 0i = -1$$

$$e^{2+i\pi} = e^2.e^{i\pi} = e^2\big(\cos(\pi) + i\sin(\pi)\big) = e^2(-1+0i) = -e^2$$

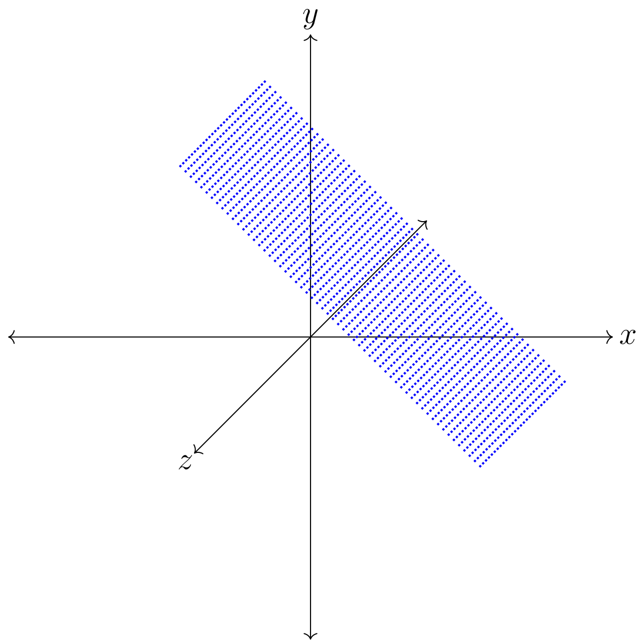
$\square$

Surfaces and Vectors

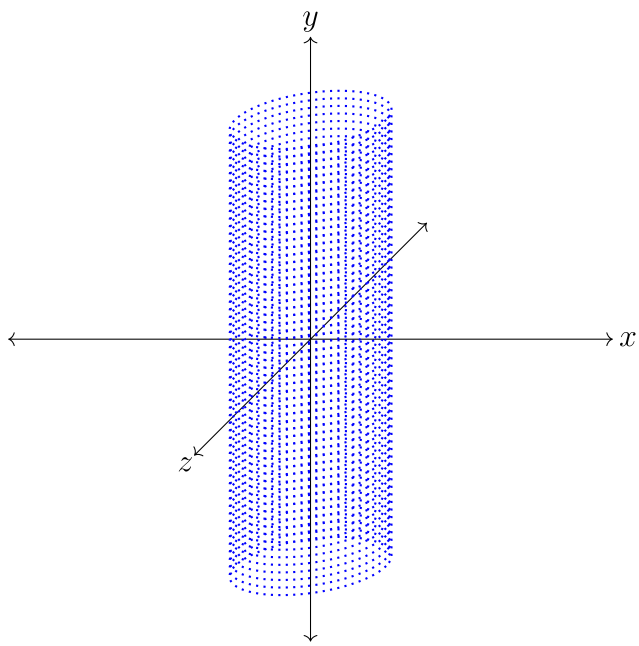
Problem 1: Describe and sketch the surface in $\mathbb{R}^3$ represented by the following equations:-

1. $x + y = 2$
2. $x^2 + z^2 = 9$
3. $x^2 + y^2 + z^2 - 2x - 2z - 2 = 0$

Solution. (1) $x + y = 2$ is the equation of a plane that intersects the x-axis at $(2, 0, 0)$, the y-axis at $(0, 2, 0)$ and does not intersect the z-axis at all.



(2) $x^2 + z^2 = 9$ is the equation of a cylinder of radius 3 and axis being the y-axis.

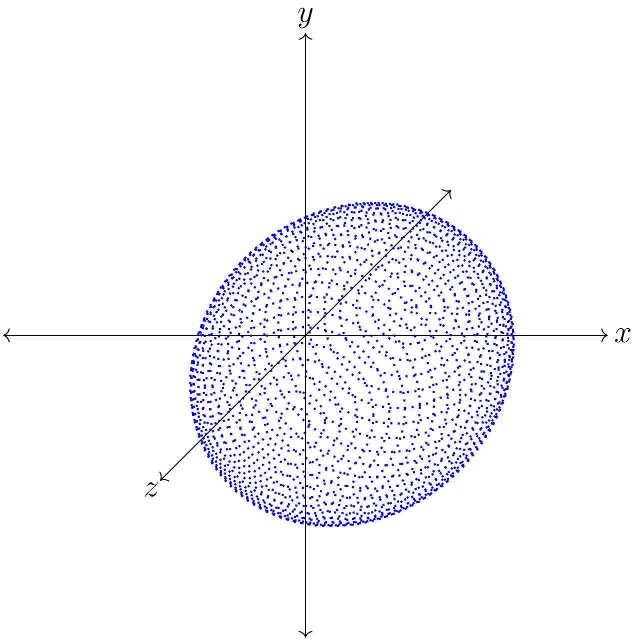


(3) Use completion of squares to bring in standard form.

$$x^2 + y^2 + z^2 - 2x - 2z - 2 = 0 \Rightarrow (\underbrace{x^2 - 2x + 1}_{(x-1)^2} - 1) + y^2 + (\underbrace{z^2 - 2z + 1}_{(z-1)^2} - 1) - 2 = 0$$

$$\Rightarrow (x-1)^2 + y^2 + (z-1)^2 = 4$$

This is the equation of a sphere with radius 2 and center at $(1, 0, 1)$.



□

Problem 2: Find the equation of a sphere centered at $(0, 0, 1)$ and passing through the origin.

Solution. Since the sphere passes through origin, the distance between origin and the center $(0, 0, 1)$ is the radius. Therefore,

$$r = \sqrt{(0-0)^2 + (0-0)^2 + (1-0)^2} = 1$$

Then the equation of the given sphere is

$$(x-0)^2 + (y-0)^2 + (z-1)^2 = 1^2 \Rightarrow x^2 + y^2 + z^2 - 2z + 1 = 1$$

which is

$$\boxed{x^2 + y^2 + z^2 - 2z = 0}$$

□

Problem 3: Let $\vec{a} = 4\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b}$ be the vector from $A(0, 3, 1)$ to $B(2, 3, -1)$.

1. Find the components of $\vec{b}$ and write it in the form $x\hat{i} + y\hat{j} + z\hat{k}$.
2. Find $4\vec{a} - 3\vec{b}$ and $|\vec{a} - \vec{b}|$.
3. Find the vector that has the same direction as $\vec{b}$ but has length 4.
4. Find the unit vector in the direction of $\vec{b} - \vec{a}$.

Solution. (1) $\vec{b} = (2-0)\hat{i} + (3-3)\hat{j} + (-1-1)\hat{k} = 2\hat{i} - 2\hat{k}$

Therefore, the components are $b_x = 2$, $b_y = 0$, $b_z = -2$.

(2)

$$4\vec{a} - 3\vec{b} = 4(4\hat{i} + 3\hat{j} - \hat{k}) - 3(2\hat{i} - 2\hat{k}) = 16\hat{i} + 12\hat{j} - 4\hat{k} - 6\hat{i} + 6\hat{k} = 10\hat{i} + 12\hat{j} + 2\hat{k}$$

$$\vec{a} - \vec{b} = 4\hat{i} + 3\hat{j} - \hat{k} - (2\hat{i} - 2\hat{k}) = 2\hat{i} + 3\hat{j} + \hat{k} \Rightarrow |\vec{a} - \vec{b}| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$$

(3) The vector with length 4 and direction same as $\vec{b}$ is 4 times the unit vector in the direction of $\vec{b}$. Thus, such a vector is given by

$$4\frac{\vec{b}}{|\vec{b}|} = 4\frac{2\hat{i} - 2\hat{k}}{\sqrt{(2)^2 + (-2)^2}} = 4\frac{2\hat{i} - 2\hat{k}}{\sqrt{8}} = \frac{4}{\sqrt{8}}(2\hat{i} - 2\hat{k}) = \frac{4}{2\sqrt{2}}(2\hat{i} - 2\hat{k}) = \frac{4}{\sqrt{2}}\hat{i} - \frac{4}{\sqrt{2}}\hat{k} = 2\sqrt{2}\,\hat{i} - 2\sqrt{2}\,\hat{k}$$

(4)

$$\vec{b} - \vec{a} = 2\hat{i} - 2\hat{k} - (4\hat{i} + 3\hat{j} - \hat{k}) = -2\hat{i} - 3\hat{j} - \hat{k} \Rightarrow |\vec{b} - \vec{a}| = \sqrt{(-2)^2 + (-3)^2 + (-1)^2} = \sqrt{14}$$

The unit vector in the direction of $\vec{b} - \vec{a}$ is then given by

$$\frac{\vec{b} - \vec{a}}{|\vec{b} - \vec{a}|} = \frac{-2\hat{i} - 3\hat{j} - \hat{k}}{\sqrt{14}} = -\frac{2}{\sqrt{14}}\,\hat{i} - \frac{3}{\sqrt{14}}\,\hat{j} - \frac{1}{\sqrt{14}}\,\hat{k}$$

□

Problem 4: Let $\vec{a} = \hat{i} + \hat{j}$ and $\vec{b} = \hat{k} - \hat{j}$.

1. Compute $\vec{a}.\vec{b}$ and find the angle between $\vec{a}$ and $\vec{b}$.
2. Find the direction cosines and direction angles of the vector $\vec{a} - \vec{b}$.
3. Find the scalar and vector projections of $\vec{a} + \vec{b}$ onto $\vec{b}$.
4. Find the unit vector orthogonal to $\vec{a}$ and parallel to $\vec{b}$.

Solutions. (1) $\vec{a}.\vec{b} = a_x b_x + a_y b_y + a_z b_z = (1)(0) + (1)(-1) + (0)(1) = -1$

$|\vec{a}| = \sqrt{(1)^2 + (1)^2 + (0)^2} = \sqrt{2}$ and $|\vec{b}| = \sqrt{(0)^2 + (-1)^2 + (1)^2} = \sqrt{2}$

The angle θ between $\vec{a}$ and $\vec{b}$ is given by

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \frac{-1}{\sqrt{2}\,\sqrt{2}} = -\frac{1}{2} \Rightarrow \theta = \pi - \cos^{-1}\left(\frac{1}{2}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

(2) The direction cosines of a vector $\vec{p}$ are given by

$$\cos\alpha = \frac{p_x}{|\vec{p}|} \quad , \quad \cos\beta = \frac{p_y}{|\vec{p}|} \quad , \quad \cos\gamma = \frac{p_z}{|\vec{p}|}$$

Now, $\vec{a} - \vec{b} = (\hat{i} + \hat{j}) - (\hat{k} - \hat{j}) = \hat{i} + 2\hat{j} - \hat{k}$ and $|\vec{a} - \vec{b}| = \sqrt{(1)^2 + (2)^2 + (-1)^2} = \sqrt{6}$.

Therefore, the direction cosines of $\vec{a} - \vec{b}$ are given by

$$\cos\alpha = \frac{1}{\sqrt{6}} \quad , \quad \cos\beta = \frac{2}{\sqrt{6}} \quad , \quad \cos\gamma = \frac{-1}{\sqrt{6}}$$

Then the direction angles would be

$$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{6}}\right) \quad , \quad \beta = \cos^{-1}\left(\frac{2}{\sqrt{6}}\right) \quad , \quad \gamma = \pi - \cos^{-1}\left(\frac{1}{\sqrt{6}}\right)$$

(3) The scalar projection of a vector $\vec{p}$ onto a vector $\vec{q}$ is given by

$$\text{comp}_{\vec{q}}\vec{p} = \vec{p}.\hat{q}$$

where $\hat{q}$ is the unit vector in the direction of $\vec{q}$. The vector projection of $\vec{p}$ onto $\vec{q}$ is given by

$$\text{proj}_{\vec{q}}\vec{p} = (\vec{p}.\hat{q})\hat{q}$$

Now $\vec{p} = \vec{a} + \vec{b} = (\hat{\imath} + \hat{\jmath}) + (\hat{k} - \hat{\jmath}) = \hat{\imath} + \hat{k}$ and $\hat{q} = \dfrac{\vec{b}}{|\vec{b}|} = \dfrac{1}{\sqrt{2}}(\hat{k} - \hat{\jmath})$.

The scalar projection of $\vec{a} + \vec{b}$ onto $\vec{b}$ is then given by

$$\vec{p}.\hat{q} = (\hat{\imath} + \hat{k}).\frac{1}{\sqrt{2}}(\hat{k} - \hat{\jmath}) = \frac{1}{\sqrt{2}}\big((1)(0) + (0)(-1) + (1)(1)\big) = \frac{1}{\sqrt{2}}$$

The vector projection of $\vec{a} + \vec{b}$ onto $\vec{b}$ is then given by

$$(\vec{p}.\hat{q})\hat{q} = \frac{1}{\sqrt{2}}\hat{q} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}(\hat{k} - \hat{\jmath}) = \frac{1}{2}(\hat{k} - \hat{\jmath}).$$

(4) Let $\vec{p}$ be the unit vector orthogonal to $\vec{a}$ and parallel to $\vec{b}$.

Since $\vec{p}$ is parallel to $\vec{b}$, it has to be proportional to $\vec{b}$. Therefore,

$$\vec{p} = c\,\hat{k} - c\,\hat{\jmath}$$

for some scalar $c \in \mathbb{R}$.

Since $\vec{p}$ is orthogonal to $\vec{a}$, we must have $\vec{p}.\vec{a} = 0$. Therefore,

$$(c\,\hat{k} - c\,\hat{\jmath}).(\hat{\imath} + \hat{\jmath}) = 0 \Rightarrow (0)(1) + (-c)(1) + (c)(0) = 0 \Rightarrow -c = 0 \Rightarrow c = 0$$

Thus, we must have $\vec{p} = \vec{0}$, but then $|\vec{p}| = \sqrt{0^2 + 0^2 + 0^2} = 0$ and $\vec{p}$ cannot be a unit vector.

Hence, there is no such vector which is orthogonal to $\vec{a}$ and parallel to $\vec{b}$. $\square$

Problem 5: Find the following vectors, without using determinant, but by using the properties of cross products.

1. $(\hat{\imath} \times \hat{\jmath}) \times \hat{k}$
2. $(\hat{\imath} + 2\hat{\jmath}) \times (\hat{\imath} - \hat{\jmath} + 2\hat{k})$

Solutions. (1) We know that $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$. Therefore,

$$(\hat{\imath} \times \hat{\jmath}) \times \hat{k} = -\hat{k} \times (\hat{\imath} \times \hat{\jmath}) = -\big((\hat{k}.\hat{\jmath})\hat{\imath} - (\hat{k}.\hat{\imath})\hat{\jmath}\big) = -(0\hat{\imath} - 0\hat{\jmath}) = \vec{0}$$

Alternatively, since $\hat{\imath} \times \hat{\jmath} = \hat{k}$, we have

$$(\hat{\imath} \times \hat{\jmath}) \times \hat{k} = \hat{k} \times \hat{k} = \vec{0}.$$

(2) We use distributivity of cross product over addition.

$$\begin{aligned}(\hat{\imath} + 2\hat{\jmath}) \times (\hat{\imath} - \hat{\jmath} + 2\hat{k}) &= \hat{\imath} \times (\hat{\imath} - \hat{\jmath} + 2\hat{k}) + 2\hat{\jmath} \times (\hat{\imath} - \hat{\jmath} + 2\hat{k}) \\ &= \hat{\imath} \times \hat{\imath} - \hat{\imath} \times \hat{\jmath} + 2\,\hat{\imath} \times \hat{k} + 2\,\hat{\jmath} \times \hat{\imath} - 2\,\hat{\jmath} \times \hat{\jmath} + 4\,\hat{\jmath} \times \hat{k} \\ &= \vec{0} - \hat{k} + 2\,(-\hat{\jmath}) + 2\,(-\hat{k}) - 2(\vec{0}) + 4\,\hat{\imath} = \boxed{4\,\hat{\imath} - 2\,\hat{\jmath} - 3\,\hat{k}}\end{aligned}$$

□

Problem 6: Let $P(0, -2, 0)$, $Q(4, 1, -2)$, $R(5, 3, 1)$ be points in the 3-D space.

1. Find the area of the triangle PQR.
2. Find a nonzero vector orthogonal to the plane passing through points P, Q and R.

Solutions. (1) The area of the triangle PQR is given by

$$A = \frac{1}{2}\left|\vec{a} \times \vec{b}\right|$$

where $\vec{a}$ is vector from Q to P and $\vec{b}$ is the vector from Q to R.
Note that $\vec{a}$ and $\vec{b}$ can be chosen to be any two adjacent sides of the triangle PQR.

Now, $\vec{a} = \big(4 - 0\big)\,\hat{\imath} + \big(1 - (-2)\big)\,\hat{\jmath} + \big(-2 - 0\big)\,\hat{k} = 4\,\hat{\imath} + 3\,\hat{\jmath} - 2\,\hat{k}$ and

$\vec{b} = \big(5 - 4\big)\,\hat{\imath} + \big(3 - 1\big)\,\hat{\jmath} + \big(1 - (-2)\big)\,\hat{k} = \hat{\imath} + 2\,\hat{\jmath} + 3\,\hat{k}$.

The cross product $\vec{a} \times \vec{b}$ is given by

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & 3 & -2 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ 2 & 3 \end{vmatrix}\,\hat{\imath} - \begin{vmatrix} 4 & -2 \\ 1 & 3 \end{vmatrix}\,\hat{\jmath} + \begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix}\,\hat{k} = 13\,\hat{\imath} - 14\,\hat{\jmath} + 5\,\hat{k}$$

Thus, the ares of triangle PQR is given by

$$A = \frac{1}{2}\sqrt{(13)^2 + (-14)^2 + (5)^2} = \frac{1}{2}\sqrt{390}$$

(2) The nonzero vector orthogonal to the plane passing through P, Q, R is proportional to $\vec{a} \times \vec{b}$ where $\vec{a}$ is vector from Q to P and $\vec{b}$ is the vector from Q to R. As in the previous part $\vec{a}$ and $\vec{b}$ can be chosen to be any two adjacent sides of the triangle PQR.

Thus, one such vector is $\vec{a} \times \vec{b} = 13\,\hat{\imath} - 14\,\hat{\jmath} + 5\,\hat{k}$. □

Problem 7: Find the volume of the parallelepiped determined by the vectors

$$\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\vec{b} = -\hat{i} + \hat{j} + 2\hat{k}$$

$$\vec{c} = 2\hat{i} + \hat{j} + 4\hat{k}$$

Solutions. The volume of the parallelepiped determined by any three given vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ is given by $|(\vec{a} \times \vec{b}).\vec{c}|$. So, the required volume is

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{vmatrix} .(2\,\hat{i} + \hat{j} + 4\,\hat{k}) = \begin{vmatrix} 2 & 1 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} + 4\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix}$$

$$= 2(1) - 1(5) + 4(3) = 9$$

$\square$

Lines and Planes

Problem 1: Find the vector, parametric and Cartesian (also called symmetric) equation of the following lines:-

1. The line that passes through the points $(-8, 1, 4)$ and $(3, -2, 4)$.

   Solution: The vector equation is $\vec{r}(t) = \vec{a} + t\vec{v}$, $t \in \mathbb{R}$, where $\vec{a}$ is the position vector of one of the points and $\vec{v}$ is the direction vector of the line.

   $$\vec{v} = \big(3 - (-8)\big)\hat{\imath} + \big(-2-1\big)\hat{\jmath} + \big(4-4\big)\hat{k} = 11\,\hat{\imath} - 3\,\hat{\jmath}$$

   Choose the point $(-8, 1, 4)$ for $\vec{a}$. Then

   $$\vec{a} = -8\,\hat{\imath} + \hat{\jmath} + 4\,\hat{k}$$

   so that

   $$\vec{r}(t) = -8\,\hat{\imath} + \hat{\jmath} + 4\,\hat{k} + t(11\,\hat{\imath} - 3\,\hat{\jmath}), \quad t \in \mathbb{R}$$

   Thus, the vector equation of the given line is

   $$\boxed{\vec{r}(t) = (-8+11t)\hat{\imath} + (1-3t)\hat{\jmath} + 4\hat{k}, \quad t \in \mathbb{R}}$$

   The parametric equation will then be

   $$\boxed{x(t) = -8 + 11t, \quad y(t) = 1 - 3t, \quad z(t) = 4, \quad t \in \mathbb{R}}$$

   And the cartesian equation would be

   $$\boxed{\frac{x+8}{11} = \frac{y-1}{-3}, \qquad z = 4}$$

2. The line that passes through the point $(2, 1, 0)$ and is perpendicular to both $\hat{\imath} + \hat{\jmath}$ and $\hat{\jmath} + \hat{k}$.

   Solution: We have one point, that is, $\vec{a} = 2\,\hat{\imath} + \hat{\jmath}$ and direction will be given by the vector that is perpendicular to both $\hat{\imath} + \hat{\jmath}$ and $\hat{\jmath} + \hat{k}$. So,

   $$\vec{v} = \big(\hat{\imath} + \hat{\jmath}\big) \times \big(\hat{\jmath} + \hat{k}\big) = \big(\hat{\imath} \times \hat{\jmath}\big) + \big(\hat{\imath} \times \hat{k}\big) + \big(\hat{\jmath} \times \hat{\jmath}\big) + \big(\hat{\jmath} \times \hat{k}\big) = \hat{k} - \hat{\jmath} + \vec{0} + \hat{\imath}$$

   Thus, $\vec{v} = \hat{\imath} - \hat{\jmath} + \hat{k}$ and we have

   $$\vec{r}(t) = \vec{a} + t\vec{v} = 2\,\hat{\imath} + \hat{\jmath} + t(\hat{\imath} - \hat{\jmath} + \hat{k}) = (2+t)\,\hat{\imath} + (1-t)\,\hat{\jmath} + t\,\hat{k}, \quad t \in \mathbb{R}$$

   Thus, the vector equation of the given line is

   $$\boxed{\vec{r}(t) = (2+t)\,\hat{\imath} + (1-t)\,\hat{\jmath} + t\,\hat{k}, \quad t \in \mathbb{R}}$$

   The parametric equation will then be

   $$\boxed{x(t) = 2 + t, \quad y(t) = 1 - t, \quad z(t) = t, \quad t \in \mathbb{R}}$$

   And the cartesian equation would be

   $$\boxed{x - 2 = 1 - y = z}$$

3. The line that passes through the point $(-6, 2, 3)$ and is parallel to the line $\dfrac{x}{2} = \dfrac{y}{3} = \dfrac{z+1}{1}$.

   Solution: The required line passes through $\vec{a} = -6\,\hat{\imath} + 2\,\hat{\jmath} + 3\,\hat{k}$ and since it is parallel to the line $\dfrac{x}{2} = \dfrac{y}{3} = \dfrac{z+1}{1}$, its direction is same as that of $\dfrac{x}{2} = \dfrac{y}{3} = \dfrac{z+1}{1}$.
   Hence, we have $\vec{v} = 2\,\hat{\imath} + 3\,\hat{\jmath} + \hat{k}$. So that

   $$\vec{r}(t) = \vec{a} + t\vec{v} = -6\,\hat{\imath} + 2\,\hat{\jmath} + 3\,\hat{k} + t(2\,\hat{\imath} + 3\,\hat{\jmath} + \hat{k}) = (-6+2t)\,\hat{\imath} + (2+3t)\,\hat{\jmath} + (3+t)\,\hat{k}$$

   Thus, the vector equation of the given line is

   $$\boxed{\vec{r}(t) = (-6+2t)\,\hat{\imath} + (2+3t)\,\hat{\jmath} + (3+t)\,\hat{k}, \quad t \in \mathbb{R}}$$

   The parametric equation will then be

   $$\boxed{x(t) = -6+2t, \quad y(t) = 2+3t, \quad z(t) = 3+t, \quad t \in \mathbb{R}}$$

   And the Cartesian equation would be

   $$\boxed{\frac{x+6}{2} = \frac{y-2}{3} = z-3}$$

4. The line that passes through the point $(1, 0, 6)$ and is perpendicular to the plane $x + 3y + z = 5$.

   Solution: The required line passes through $\vec{a} = \hat{\imath} + 6\,\hat{k}$ and since it is perpendicular to the plane $x + 3y + z = 5$, its direction is same as the normal vector of $x + 3y + z = 5$. Hence, $\vec{v} = \hat{\imath} + 3\,\hat{\jmath} + \hat{k}$ and we have

   $$\vec{r}(t) = \hat{\imath} + 6\,\hat{k} + t(\hat{\imath} + 3\,\hat{\jmath} + \hat{k}) = (1+t)\,\hat{\imath} + 3t\,\hat{\jmath} + (6+t)\,\hat{k}$$

   Thus, the vector equation of the given line is

   $$\boxed{\vec{r}(t) = (1+t)\,\hat{\imath} + 3t\,\hat{\jmath} + (6+t)\,\hat{k}, \quad t \in \mathbb{R}}$$

   The parametric equation will then be

   $$\boxed{x(t) = 1+t, \quad y(t) = 3t, \quad z(t) = 6+t, \quad t \in \mathbb{R}}$$

   And the Cartesian equation would be

   $$\boxed{x-1 = \frac{y}{3} = z-6}$$

Problem 2: Determine whether the lines L_1 and L_2 are parallel, skew or intersecting. If they intersect, find the point of intersection.

1. $$L_1 : \frac{x-3}{2} = \frac{y-4}{-1} = \frac{z-1}{3} \quad ; \quad L_2 : \frac{x-1}{4} = \frac{y-3}{-2} = \frac{z-4}{5}$$

   Solution: The direction of L_1 is
   $$\vec{v}_1 = 2\,\hat{i} - \hat{j} + 3\,\hat{k}$$
   and the direction of L_2 is
   $$\vec{v}_2 = 4\,\hat{i} - 2\,\hat{j} + 5\,\hat{k}$$
   If L_1, L_2 were parallel, we would have some scalar $\alpha \in \mathbb{R}$ such that $\vec{v}_1 = \alpha\,\vec{v}_2$, so that
   $$2 = 4\alpha \Rightarrow \alpha = \frac{1}{2}$$
   $$-1 = -2\,\alpha \Rightarrow \alpha = \frac{1}{2}$$
   $$3 = 5\alpha \Rightarrow \alpha = \frac{3}{5}$$
   which is inconsistent. Therefore, L_1 and L_2 are not parallel.

   Now in parametric form we have

   $L_1 : x(t) = 3{+}2t,\ y(t) = 4{-}t,\ z(t) = 1{+}3t$ and $L_2 : x(s) = 1{+}4s,\ y(s) = 3{-}2s,\ z(s) = 4{+}5s$

   Equating x, y and z, we get
   $$3 + 2t = 1 + 4s \ldots\ldots (1)$$
   $$4 - t = 3 - 2s \ldots\ldots (2)$$
   $$1 + 3t = 4 + 5s \ldots\ldots (3)$$
   Solving (1) and (2) for t and s we have:- $t = 4 - 3 + 2s = 1 + 2s$ from (2). Substituting this for t in (1) we get $3 + 2(1 + 2s) = 1 + 4s \Rightarrow 3 + 2 + 4s = 1 + 4s \Rightarrow 5 = 1$ which is not possible for any value of t and s. Therefore, the given lines have to be $\boxed{skew}$.

2. $$L_1 : x = 5 - 12t, \quad y = 3 + 9t, \quad z = 1 - 3t \quad ; \quad L_2 : x = 3 + 8s, \quad y = -6s, \quad z = 7 + 2s$$

   Solution: The direction of L_1 is
   $$\vec{v}_1 = -12\,\hat{i} + 9\,\hat{j} - 3\,\hat{k}$$
   and the direction of L_2 is
   $$\vec{v}_2 = 8\,\hat{i} - 6\,\hat{j} + 2\,\hat{k}$$

If L_1, L_2 were parallel, we would have some scalar $\alpha \in \mathbb{R}$ such that $\vec{v}_1 = \alpha\,\vec{v}_2$, so that

$$-12 = 8\alpha \Rightarrow \alpha = -\frac{3}{2}$$

$$9 = -6\,\alpha \Rightarrow \alpha = -\frac{3}{2}$$

$$-3 = 2\alpha \Rightarrow \alpha = -\frac{3}{2}$$

which is consistent. Therefore, the given lines L_1 and L_2 are $\boxed{parallel}$.

3.
$$L_1 : \frac{x-2}{1} = \frac{y-3}{-2} = \frac{z-1}{-3} \quad ; \quad L_2 : \frac{x-3}{1} = \frac{y+4}{3} = \frac{z-2}{-7}$$

Solution: The direction of L_1 is

$$\vec{v}_1 = \hat{\imath} - 2\,\hat{\jmath} - 3\,\hat{k}$$

and the direction of L_2 is

$$\vec{v}_2 = \hat{\imath} + 3\,\hat{\jmath} - 7\,\hat{k}$$

If L_1, L_2 were parallel, we would have some scalar $\alpha \in \mathbb{R}$ such that $\vec{v}_1 = \alpha\,\vec{v}_2$, so that

$$1 = \alpha \Rightarrow \alpha = 1$$

$$-2 = 3\,\alpha \Rightarrow \alpha = -\frac{2}{3}$$

$$-3 = -7\alpha \Rightarrow \alpha = \frac{3}{7}$$

which is inconsistent. Therefore, L_1 and L_2 are not parallel.

Now, in parametric form, we have

$$L_1 : \; x = 2 + t,\; y = 3 - 2t,\; z = 1 - 3t \quad \text{and} \quad L_2 : \; x = 3 + s,\; y = -4 + 3s,\; z = 2 - 7s$$

Equating x, y and z we have

$$2 + t = 3 + s \ldots\ldots (1)$$

$$3 - 2t = -4 + 3s \ldots\ldots (2)$$

$$1 - 3t = 2 - 7s \ldots\ldots (3)$$

Solving (1) and (2) for t and s we have:- $t = 1 + s$ from (1). Substituting this in (2) we get $3-2(1+s) = -4+3s \Rightarrow 1-2s = -4+3s \Rightarrow 5s = 5 \Rightarrow s = 1 \Rightarrow t = 1+s = 1+1 = 2$.

So we get $t = 2$, $s = 1$ from (1) and (2). Now we plug these values of t and s in (3).

$$1 - 3(2) = 2 - 7(1) \Rightarrow 1 - 6 = 2 - 7 \Rightarrow -5 = -5$$

Therefore, the three equations are consistent and the given lines L_1 and L_2 $\boxed{intersect}$.

To find the point of intersection, either put $t = 2$ in parametric equation of L_1 or $s = 1$ in parametric equation of L_2. So we have

$$x = 2 + 2,\ y = 3 - 2(2),\ z = 1 - 3(2) \Rightarrow x = 4,\ y = -1,\ z = -5$$

Hence, the point of intersection of L_1 and L_2 is $\boxed{(4, -1, -5)}$.

Problem 3: Find the vector and Cartesian equation of the following planes.

1. The plane passing through the points $(2, 1, 2)$, $(3, -8, 6)$ and $(-2, -3, 1)$.

   Solution: From these three point we can find two vectors that lie in the plane.

   Choosing one of them to be the vector from $(2, 1, 2)$ to $(3, -8, 6)$ we get

   $$\vec{v}_1 = (3 - 2)\,\hat{\imath} + (-8 - 1)\,\hat{\jmath} + (6 - 2)\,\hat{k} = \hat{\imath} - 9\,\hat{\jmath} + 4\,\hat{k}$$

   and the other one to be the vector from $(3, -8, 6)$ to $(-2, -3, 1)$ we get

   $$\vec{v}_2 = (-2 - 3)\,\hat{\imath} + (-3 + 8)\,\hat{\jmath} + (1 - 6)\,\hat{k} = -5\,\hat{\imath} + 5\,\hat{\jmath} - 5\hat{k}$$

   Then a normal vector to the plane is given by $\vec{n} = \vec{v}_1 \times \vec{v}_2$. So we have

   $$\vec{n} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -9 & 4 \\ -5 & 5 & -5 \end{vmatrix} = 25\,\hat{\imath} - 15\,\hat{\jmath} - 40\,\hat{k}$$

   The vector equation of a plane is given by $\vec{r}.\vec{n} = \vec{a}.\vec{n}$ where $\vec{n}$ is a normal vector to the plane and $\vec{a}$ is the position vector of some point in the plane.

   Choosing the first point $(2, 1, 2)$ we have $\vec{a} = 2\,\hat{\imath} + \hat{\jmath} + 2\,\hat{k}$. Then

   $$\vec{a}.\vec{n} = 50 - 15 - 80 = -45$$

   So we have

   $$\vec{r}.(25\,\hat{\imath} - 15\,\hat{\jmath} - 40\,\hat{k}) = -45 \Rightarrow \vec{r}.(5\,\hat{\imath} - 3\,\hat{\jmath} - 8\,\hat{k}) = -9$$

   where in second step, 5 was factored out from both sides and canceled.

   Thus, the vector equation of the given plane is

   $$\boxed{\vec{r}.(5\,\hat{\imath} - 3\,\hat{\jmath} - 8\,\hat{k}) + 9 = 0}$$

   And the Cartesian equation is

   $$\boxed{5x - 3y - 8z + 9 = 0}$$

2. The plane passing through the point $(2, 0, 1)$ and perpendicular to the line $x = 3t$, $y = 2 - t$, $z = 3 + 4t$.

   Solution: For some point in the plane we have $\vec{a} = 2\,\hat{\imath} + \hat{k}$. Since the plane is perpendicular to the given line, its normal vector has to be parallel to the direction vector of this line. Now the direction vector of the given line is

   $$\vec{v} = 3\,\hat{\imath} - \hat{\jmath} + 4\,\hat{k}$$

So, a normal vector to the plane is

$$\vec{n} = 3\,\hat{i} - \hat{j} + 4\,\hat{k}$$

We have

$$\vec{a}.\vec{n} = 2(3) + 0(-1) + 1(4) = 10$$

Therefore, the vector equation of the given plane is

$$\boxed{\vec{r}.(3\,\hat{i} - \hat{j} + 4\,\hat{k}) = 10}$$

And the Cartesian equation is

$$\boxed{3x - y + 4z = 10}$$

3. The plane passing through the point $(3, -2, 8)$ and parallel to the plane $z = x + y$.

Solution: We have $\vec{a} = 3\,\hat{i} - 2\,\hat{j} + 8\,\hat{k}$. Since the plane is parallel to the plane $z = x + y$, its normal vector is parallel to that of $z = x + y$. Thus, a normal vector to the required plane can be taken to the normal vector of the plane $z = x + y$ or $x + y - z = 0$. Hence,

$$\vec{n} = \hat{i} + \hat{j} - \hat{k}$$

We have

$$\vec{a}.\vec{n} = 3(1) - 2(1) + 8(-1) = -7$$

Therefore, the vector equation of the plane is

$$\boxed{\vec{r}.(\hat{i} + \hat{j} - \hat{k}) = -7}$$

And the Cartesian equation is

$$\boxed{x + y - z = -7}$$

4. The plane that passes through the point $(3, 5, -1)$ and contains the line $x = 4 - t$, $y = 2t - 1$, $z = -3t$.

Solution: We have $\vec{a} = 3\,\hat{i} + 5\,\hat{j} - \hat{k}$. The plane contains the line

$$L : x = 4 - t \quad y = 2t - 1, \quad z = -3t$$

which has direction vector $\vec{v} = -\hat{i} + 2\,\hat{j} - 3\,\hat{k}$. We also see that L passes through the point $(4, -1, 0)$. Since the line lies in the plane this point and the direction vector $\vec{v}$ both lie in the plane. Thus, the vector joining $(3, 5, -1)$ to $(4, -1, 0)$ also lie in the plane. This vector is given by

$$\vec{w} = (4 - 3)\,\hat{i} + (-1 - 5)\,\hat{j} + (0 + 1)\,\hat{k} = \hat{i} - 6\,\hat{j} + \hat{k}$$

Now the normal vector is perpendicular to the plane and hence perpendicular to both $\vec{v}$ and $\vec{w}$ since these vector lie in the plane. Thus,

$$\vec{n} = \vec{v} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & -3 \\ 1 & -6 & 1 \end{vmatrix} = -16\,\hat{i} - 2\,\hat{j} + 4\,\hat{k}$$

$$\vec{a}.\vec{n} = 3(-16) + 5(-2) + (-1)(4) = -62$$

So, we have

$$\vec{r}.(-16\,\hat{\imath} - 2\,\hat{\jmath} + 4\,\hat{k}) = -62 \Rightarrow \vec{r}.(8\,\hat{\imath} + \hat{\jmath} - 2\,\hat{k}) = 31$$

where in the second step -2 was factored out from both the sides and cancelled. Thus the vector equation of the given plane is

$$\boxed{\vec{r}.(8\,\hat{\imath} + \hat{\jmath} - 2\,\hat{k}) = 31}$$

And the Cartesian equation is given by

$$\boxed{8x + y - 3z = 31}$$

5. The plane passing through the point $(1, 5, 1)$ and perpendicular to the planes $2x + y - 2z = 2$ and $x + 3z = 4$.

   Solution: We have $\vec{a} = \hat{\imath} + 5\,\hat{\jmath} + \hat{k}$. Since the plane is perpendicular to two given planes, its normal vector is perpendicular to the normal vectors of the two given planes $2x + y - 2z = 2$ and $x + 3z = 4$. The normal vectors of these planes are

$$\vec{n}_1 = 2\,\hat{\imath} + \hat{\jmath} - 2\,\hat{k}$$

$$\vec{n}_2 = \hat{\imath} + 3\,\hat{k}$$

   Thus, the normal vector $\vec{n}$ of the required plane is

$$\vec{n} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & -2 \\ 1 & 0 & 3 \end{vmatrix} = 3\,\hat{\imath} - 8\,\hat{\jmath} - \hat{k}$$

   Then $\vec{a}.\vec{n} = 1(3) + 5(-8) + 1(-1) = -38$.

   Therefore, the vector equation of the plane is

$$\boxed{\vec{r}.(3\,\hat{\imath} - 8\,\hat{\jmath} - \hat{k}) = -38}$$

   And the Cartesian equation is

$$\boxed{3x - 8y - z = -38}$$

Problem 4: Consider the planes $P_1 : 3x - 2y + z = 1$, $P_2 : 2x + y - 3z = 3$ and the line $L : x = 2 - 2t$, $y = -15 - t$, $z = 1 + 4t$.

1. Find the points of intersection of L with P_1 and P_2.

   Solution: <u>Point of intersection of L and P_1</u>:-

   Put $x = 2 - 2t$, $y = -15 - t$, $z = 1 + 4t$ in the equation $3x - 2y + z = 1$ of P_1. We get

$$3(2 - 2t) - 2(-15 - t) + (1 + 4t) = 1 \Rightarrow 6 - 6t + 30 + 2t + 1 + 4t = 1 \Rightarrow 37 = 1$$

   which is inconsistent. Since the coefficient of t becomes 0 on the Left hand side the line L is parallel to P_1. So it can either completely lie in the plane or not intersect P_1 at all. But if it would have been completely lying in the plane we would have got $1 = 1$ which is not the case. Therefore, $\boxed{L \text{ does not intersect } P_1}$.

Point of intersection of L and P_2:-

Put $x = 2 - 2t$, $y = -15 - t$, $z = 1 + 4t$ in the equation $2x + y - 3z = 3$ of P_2. We get

$$2(2-2t)+(-15-t)-3(1+4t) = 3 \Rightarrow 4-4t-15-t-3-12t = 3 \Rightarrow -17t = 17 \Rightarrow t = -1$$

So, the line L intersects P_2 and the point of intersection is given by

$$x = 2 - 2(-1) = 4, \quad y = -15 - (-1) = -14, \quad z = 1 + 4(-1) = -3$$

Thus, the point of intersection of L with P_2 is $\boxed{(4, -14, -3)}$.

2. Find the angle between P_1 and P_2.

   Solution: The normal vectors of P_1 and P_2 are

$$\vec{n}_1 = 3\,\hat{i} - 2\,\hat{j} + \hat{k}$$

$$\vec{n}_2 = 2\,\hat{i} + \hat{j} - 3\,\hat{k}$$

   The angle between P_1, P_2 is same as the angle between $\vec{n}_1$ and $\vec{n}_2$. Thus,

$$\cos\theta = \frac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{3(2) - 2(1) + 1(-3)}{\left(\sqrt{(3)^2 + (-2)^2 + (1)^2}\right)\left(\sqrt{(2)^2 + (1)^2 + (-3)^2}\right)} = \frac{1}{\sqrt{14}\sqrt{14}} = \frac{1}{14}$$

   Therefore, the angle between the two given planes is $\boxed{\theta = \cos^{-1}\left(\frac{1}{14}\right)}$.

3. Find the equation of the line of intersection of P_1 and P_2.

   Solution: The line of intersection of P_1 and P_2 is perpendicular to both $\vec{n}_1$ and $\vec{n}_2$. Thus, the direction vector of the line of intersection is

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & 1 \\ 2 & 1 & -3 \end{vmatrix} = 5\,\hat{i} + 11\,\hat{j} + 7\,\hat{k}$$

   To find a point lying on the line of intersection we solve the equations of two given planes for x, y, z. Since there are three variables and just two equations, we let $z = 0$ and then we get

$$3x - 2y = 1 \ldots\ldots (1)$$

$$2x + y = 3 \ldots\ldots (2)$$

   From (2) we get $y = 3-2x$. Putting this in (1) we have $3x-2(3-2x) = 1 \Rightarrow 7x-6 = 1$ $\Rightarrow 7x = 7 \Rightarrow x = 1 \Rightarrow y = 3 - 2x = 3 - 2(1) = 1$. Thus, $(1, 1, 0)$ is point lying on the line of intersection. Now we have the equation of the line of intersection to be

$$\boxed{\frac{x-1}{5} = \frac{y-1}{11} = \frac{z}{7}}$$

Quadric Surfaces

Problem 1: Reduce the following equations to one of the standard forms, classify the surface, and sketch it.

1. $4x^2 + y + 2z^2 = 0$

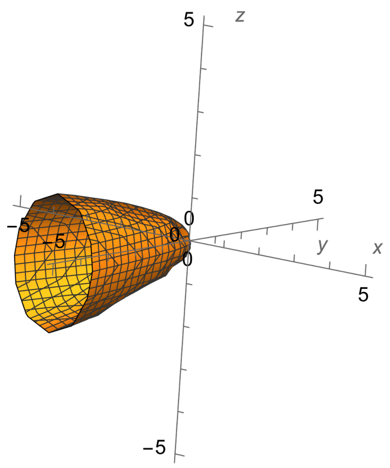
   Solution: $4x^2 + 2z^2 = -y$.

   The $x = k$ traces are: $4k^2 + 2z^2 = -y \Rightarrow 2z^2 = -(y + 4k^2)$ which are parabolas in the $x = k$ plane for any value of k. These parabolas have axis to be y-axis and they open towards the negative direction of y-axis.

   The $z = k$ traces are: $4x^2 + 2k^2 = -y \Rightarrow 4x^2 = -(y + 2k^2)$ which are parabolas in the $z = k$ plane for any value of k, having axis to be y-axis and opening towards the negative direction of y-axis.

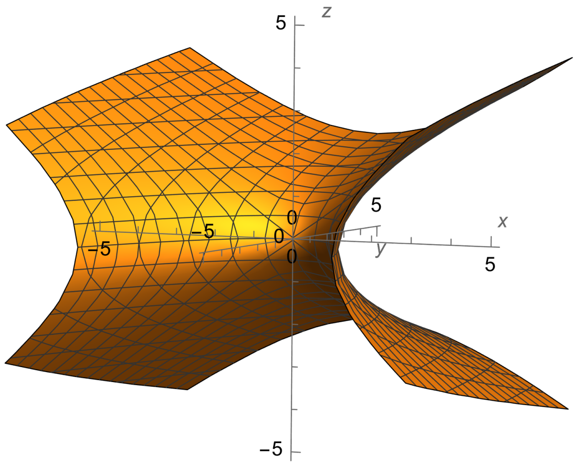
   The $y = k$ traces are: $4x^2 + 2z^2 = -k$ which are ellipses for $k < 0$.

   Thus the given equation represents an $\boxed{\text{elliptic paraboloid}}$ with axis being y-axis and opening towards the negative y-axis.



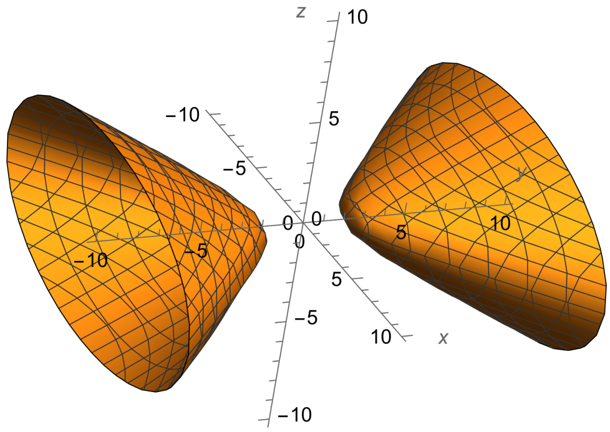
2. $x^2 + 2y - 2z^2 = 0$

   Solution: $x^2 + 2y - 2z^2 = 0 \Rightarrow 2y = 2z^2 - x^2 \Rightarrow y = z^2 - \dfrac{x^2}{2}$ which is the standard form of a $\boxed{\text{hyperbolic paraboloid}}$ with axis being the y-axis.



3. $y^2 = x^2 + 4z^2 + 4$

Solution: $y^2 = x^2 + 4z^2 + 4 \Rightarrow y^2 - x^2 - 4z^2 = 4 \Rightarrow -\dfrac{x^2}{4} + \dfrac{y^2}{4} - \dfrac{z^2}{1} = 1$ which is the standard form of a $\boxed{\text{hyperboloid with two sheets}}$ whose axis is the y-axis.



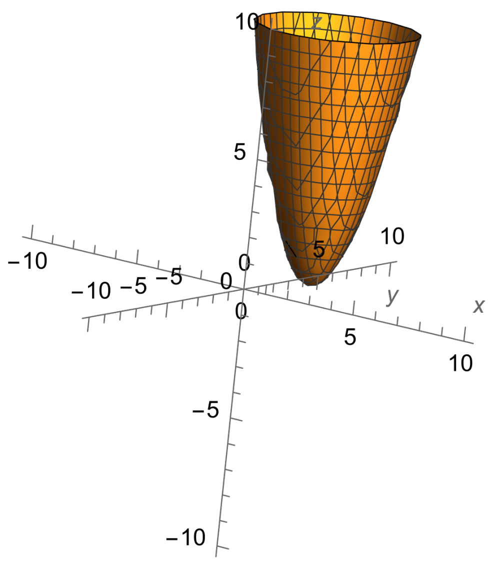
4. $x^2 + y^2 - 2x - 6y - z + 10 = 0$

Solution: $x^2 + y^2 - 2x - 6y - z + 10 = 0 \Rightarrow \underbrace{x^2 - 2x + 1}_{(x-1)^2} + \underbrace{y^2 - 6y + 9}_{(y-3)^2} - z = 0$

$\Rightarrow (x-1)^2 + (y-3)^2 - z = 0$

$$\Rightarrow z = (x-1)^2 + (y-3)^2$$

which is the standard form of an $\boxed{\text{elliptic paraboloid}}$ with vertex at $(1, 3, 0)$, axis being parallel to the z-axis and opening towards the $+$ve side of z-axis.



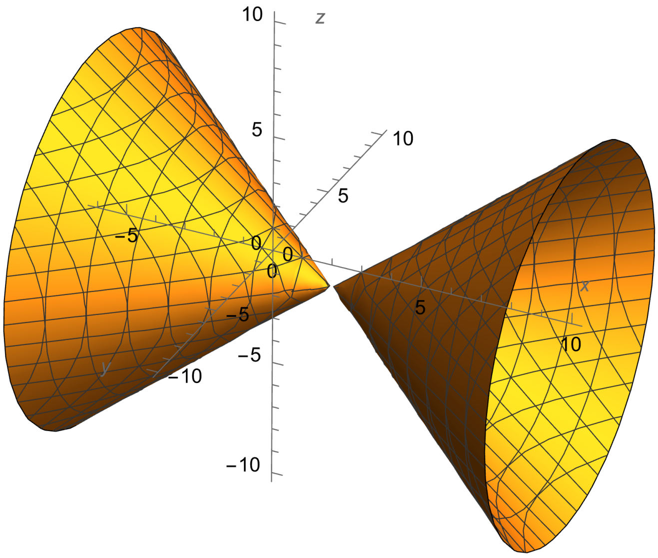
5. $x^2 - y^2 - z^2 - 4x - 2z + 3 = 0$

Solution: $x^2 - y^2 - z^2 - 4x - 2z + 3 = 0 \Rightarrow (\underbrace{x^2 - 4x + 4}_{(x-2)^2} - 4) - y^2 - (\underbrace{z^2 + 2z + 1}_{(z+1)^2} - 1) + 3 = 0$

$\Rightarrow (x-2)^2 - y^2 - (z+1)^2 = 0$

$$\Rightarrow (x-2)^2 = y^2 + (z+1)^2$$

which is standard form of a $\boxed{\text{cone}}$ centered at $(2, 0, -1)$ with axis parallel to the x-axis.



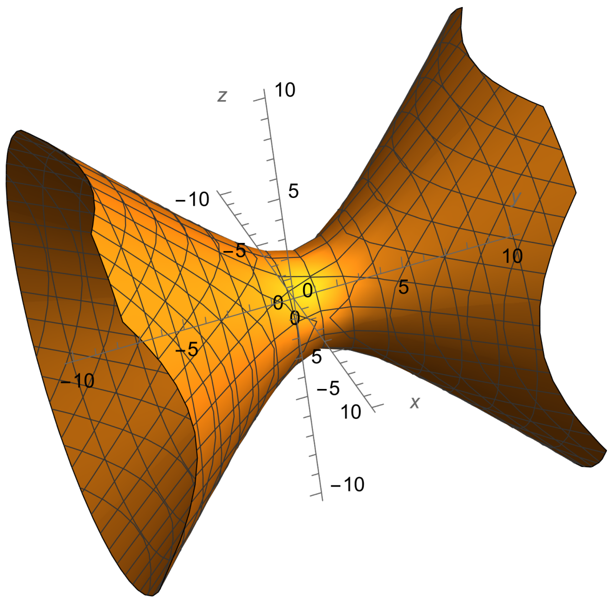
6. $x^2 - y^2 + z^2 - 4x - 2z = 0$

Solution: $x^2 - y^2 + z^2 - 4x - 2z = 0 \Rightarrow (\underbrace{x^2 - 4x + 4}_{(x-2)^2} - 4) - y^2 + (\underbrace{z^2 - 2z + 1}_{(z-1)^2} - 1) = 0$

$\Rightarrow (x-2)^2 - y^2 + (z-1)^2 = 5$

$$\Rightarrow \frac{(x-2)^2}{5} - \frac{y^2}{5} + \frac{(z-1)^2}{5} = 1$$

which is the standard form of a $\boxed{\text{hyperboloid of one sheet}}$ centered at $(2, 0, 1)$ whose axis is parallel to the y-axis.



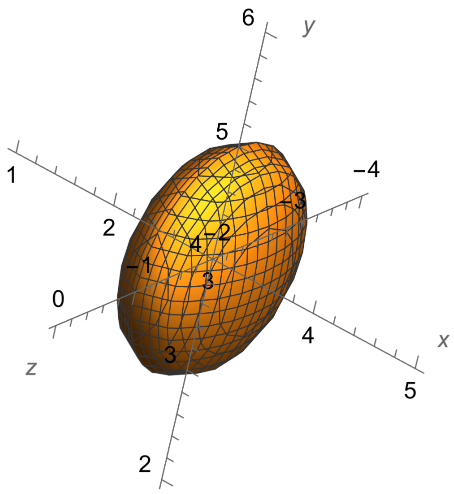
7. $4x^2 + y^2 + z^2 - 24x - 8y + 4z + 55 = 0$

Solution: $4x^2 + y^2 + z^2 - 24x - 8y + 4z + 55 = 0$

$$\Rightarrow 4(\underbrace{x^2 - 6x + 9}_{(x-3)^2} - 9) + (\underbrace{y^2 - 8y + 16}_{(y-4)^2} - 16) + (\underbrace{z^2 + 4z + 4}_{(z+2)^2} - 4) + 55 = 0$$

$$\Rightarrow 4(x-3)^2 + (y-4)^2 + (z+2)^2 = 1$$

which is the standard form of an $\boxed{\text{ellipsoid}}$ centered at $(3, 4, -2)$.

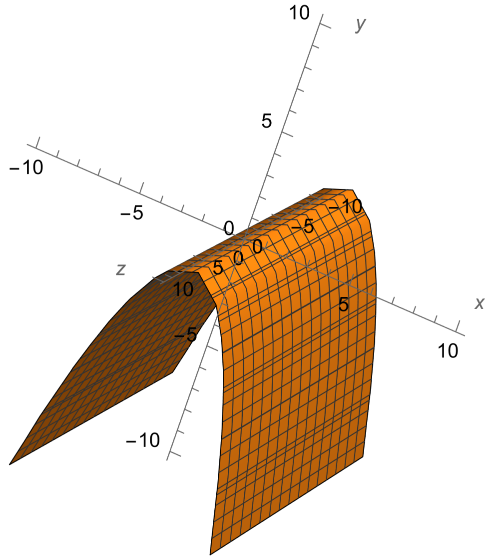


8. $x^2 - 2x + 2y - 1 = 0$

Solution: $x^2 - 2x + 2y - 1 = 0 \Rightarrow (\underbrace{x^2 - 2x + 1}_{(x-1)^2} - 1) + 2y - 1 = 0$

$$\Rightarrow (x-1)^2 = -2(y-1)$$

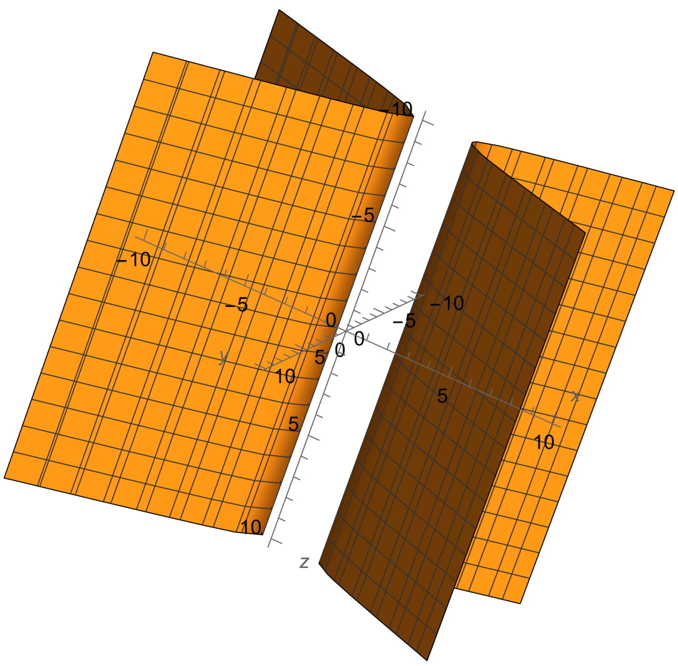
which represents a $\boxed{\text{parabolic cylinder}}$ which open toward the negative y-axis and whose axis is the line parallel to z-axis passing through $(1, 1, 0)$.



Note that if the equation was $x^2 - 2x - 2y^2 - 1 = 0$, then after completion of squares we get $(x-1)^2 - 2y^2 = 2$ or

$$\frac{(x-1)^2}{2} - \frac{y^2}{1} = 1$$

which represents a $\boxed{\text{hyperbolic cylinder}}$ whose axis is the line parallel to z-axis and passing through the point $(1, 0, 0)$.

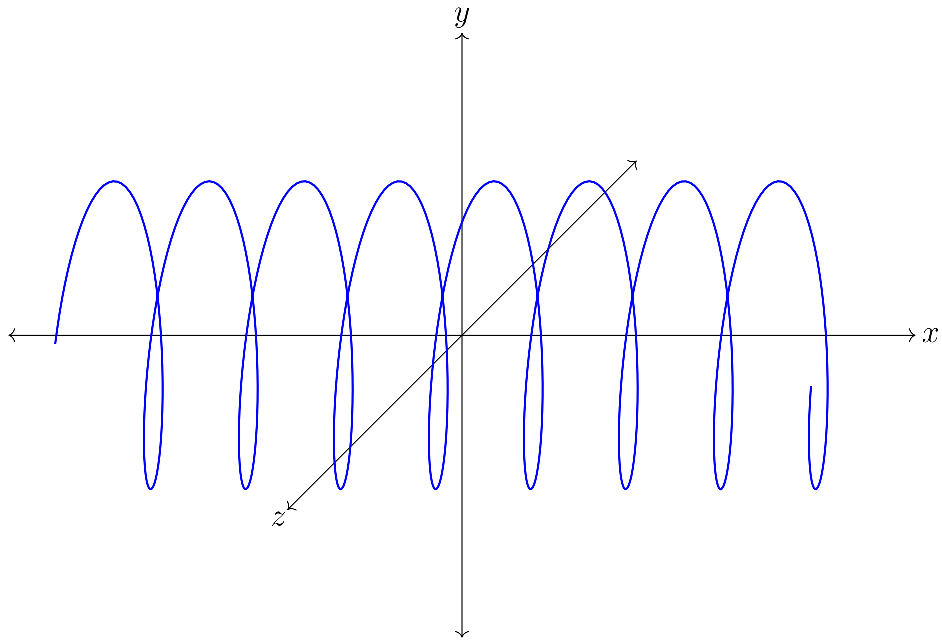


3D Curves

Problem 1: Sketch the following curves.

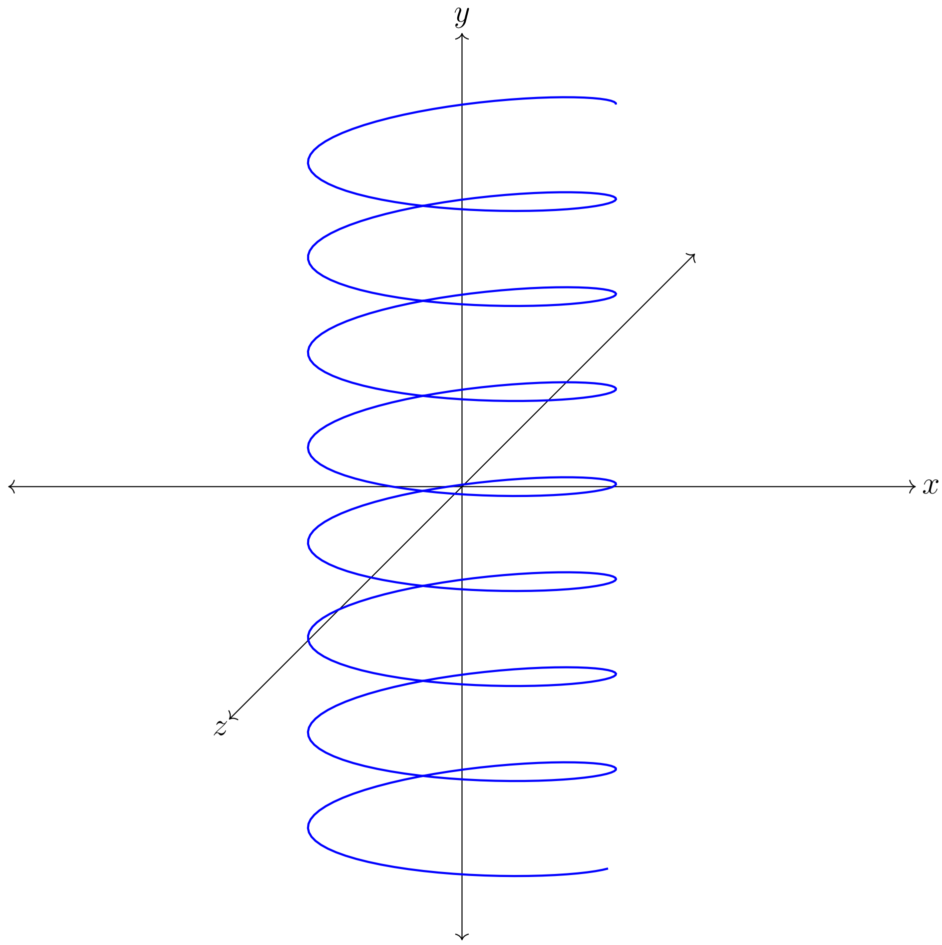
1. $\vec{r}(t) = t\,\hat{i} + 2\sin t\,\hat{j} + \cos t\,\hat{k}$

   Solution: The equation represents an elliptical helix whose axis is the x-axis



2. $\vec{r}(t) = 2\cos t\,\hat{i} + t\,\hat{j} + \sin t\,\hat{k}$

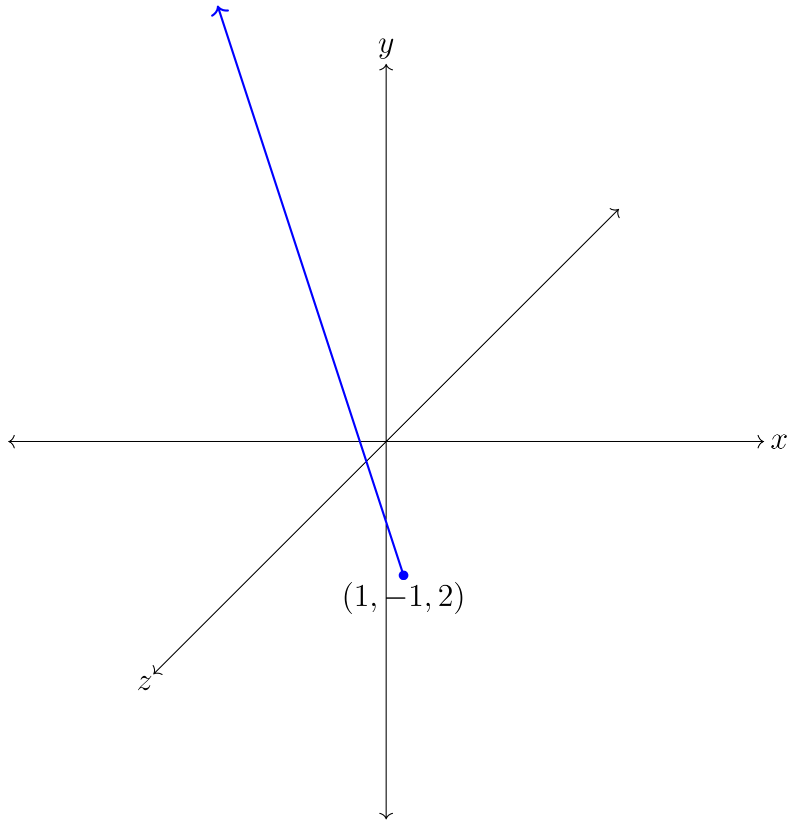
   Solution: The equation represents an elliptical helix whose axis is the y-axis.



Problem 2: Sketch the following curves.

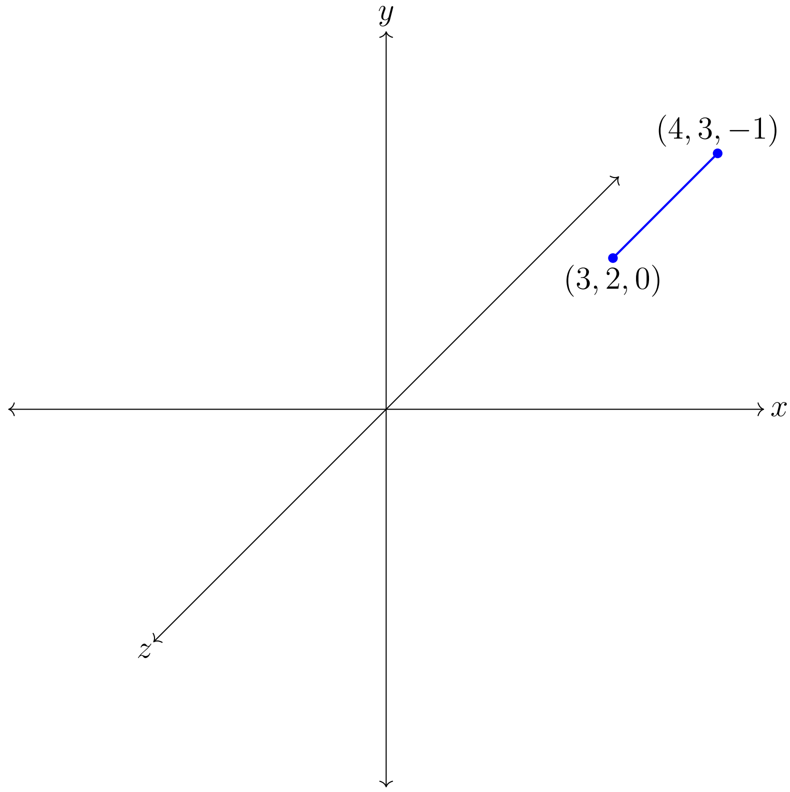
1. $\vec{r}(t) = t\,\hat{i} + (2-t)\,\hat{j} + (1+t)\,\hat{k}$, $t \le 1$

   Solution: The given curve is a ray starting from $(1, -1, 2)$ in the direction $-\hat{i} + \hat{j} - \hat{k}$.



2. $\vec{r}(t) = (2+t)\,\hat{i} + (1+t)\,\hat{j} + (1-t)\,\hat{k}$, $1 \le t \le 2$.

   Solution: The given curve is a line segment joining the points $(3, 2, 0)$ and $(4, 3, -1)$

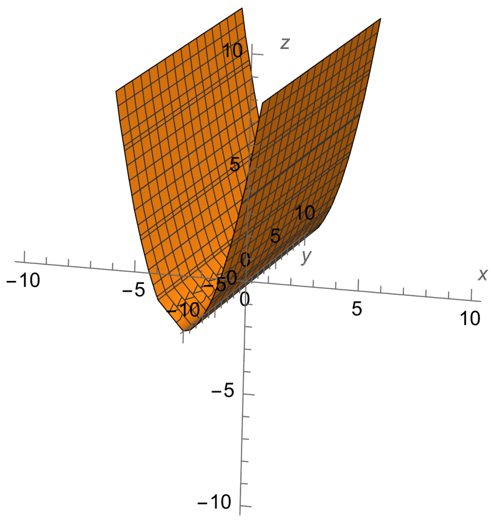


Functions of 2 Variables

Problem 1: Sketch the graphs of the following functions of two variables. Use the knowledge quadric surfaces if needed.

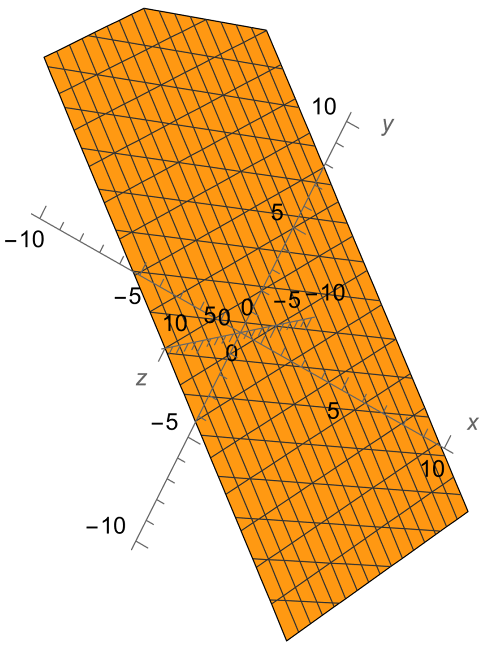
1. $f(x, y) = x^2$

   Solution: The graph is given by $z = x^2$ which is a parabolic cylinder whose axis is y-axis.



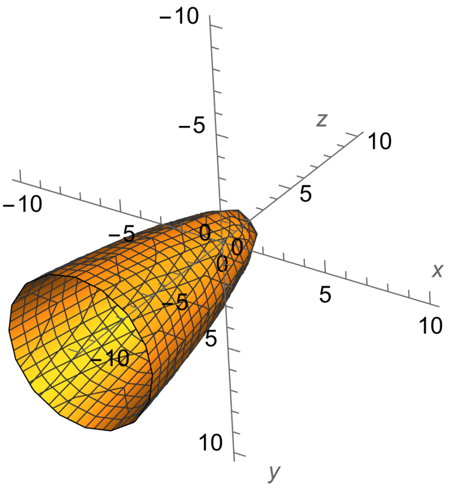
2. $f(x, y) = 10 - 4x - 5y$

   Solution: The graph is given by $z = 10 - 4x - 5y$ which is a plane.



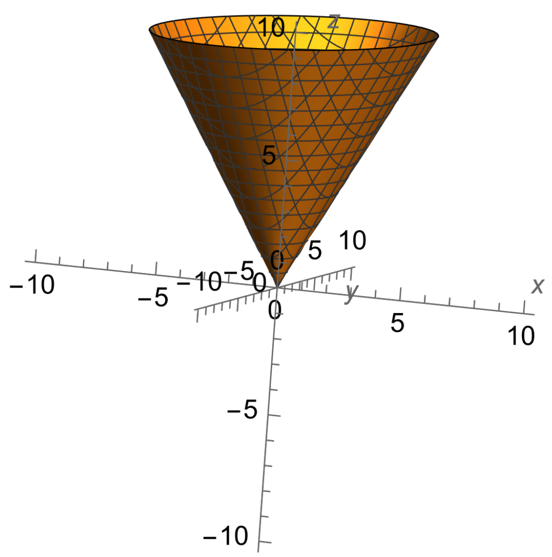
3. $f(x, y) = 2 - x^2 - y^2$

   Solution: The graph is given by $z = 2 - x^2 - y^2$ or $-(z-2) = x^2 + y^2$ which is an elliptic paraboloid with vertex at $(0, 0, 2)$ and axis being -ve z-axis.



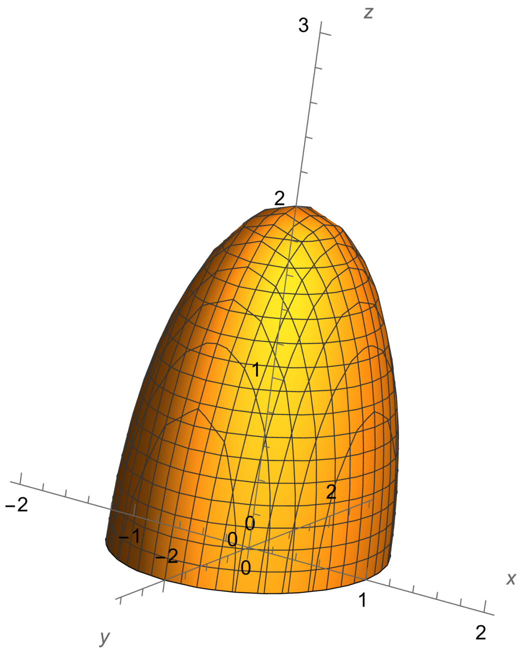
4. $f(x, y) = \sqrt{4x^2 + y^2}$

   Solution: The graph is given by $z^2 = 4x^2 + y^2$ and $z > 0$ which is the upper part of a cone with vertex at $(0, 0, 0)$ and axis being the z-axis.



5. $f(x, y) = \sqrt{4 - 4x^2 - y^2}$

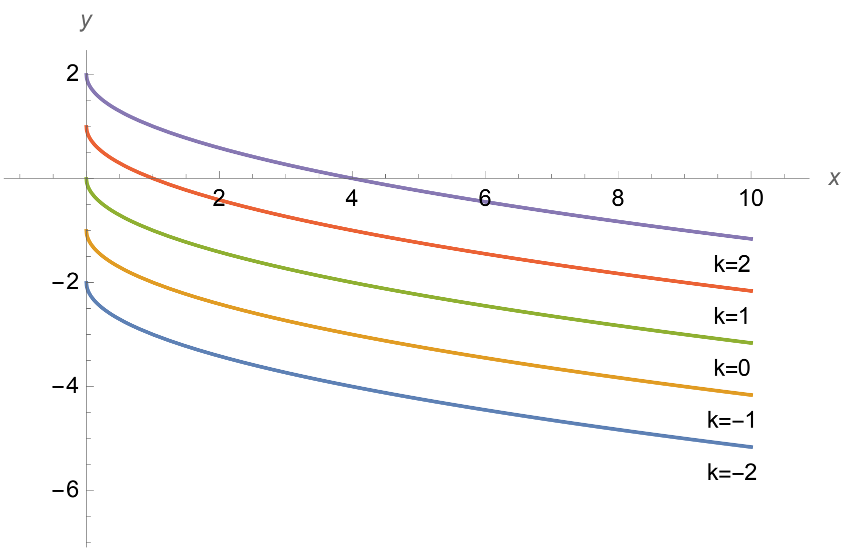
   Solution: The graph is given by $4x^2 + y^2 + z^2 = 4$ and $z > 0$ which is the upper part of an ellipsoid centered at $(0, 0, 0)$.



Problem 2: Sketch the level curves (also called contour curves) of the following functions of two variables for the level values $k = -2, -1, 0, 1, 2$.

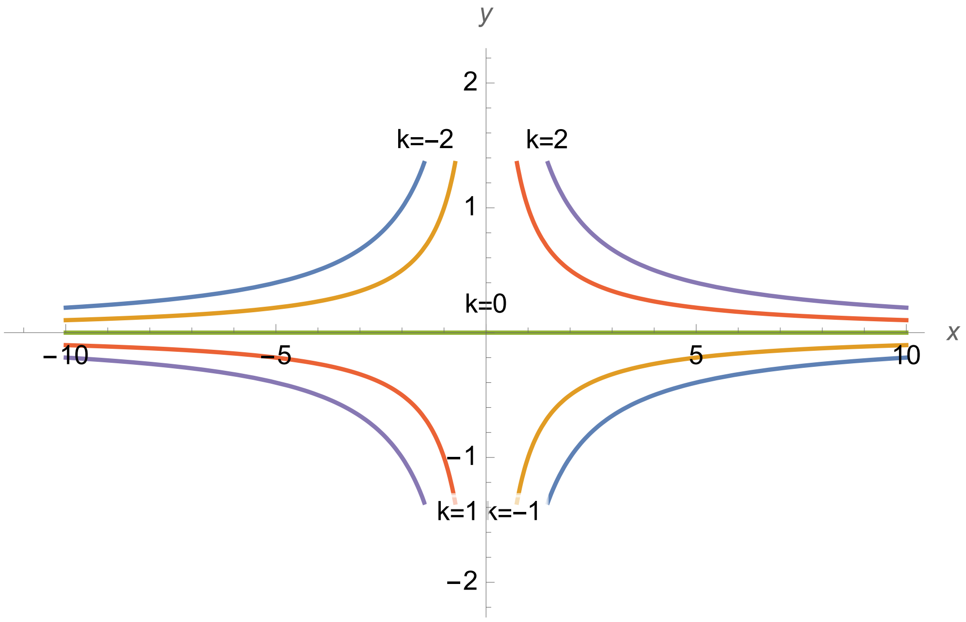
1. $f(x, y) = \sqrt{x} + y$

   Solution: The level curves are $y = k - \sqrt{x}$ for various values of k.



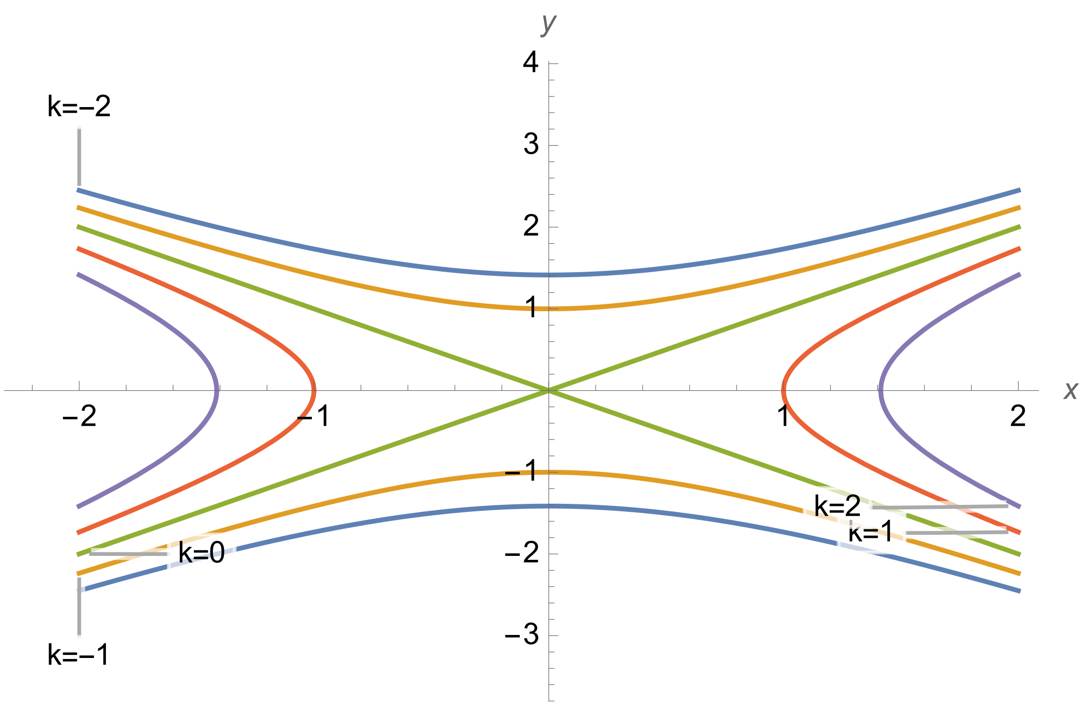
2. $f(x, y) = xy$

   Solution: The level curves are $y = \dfrac{k}{x}$ for various values of k.



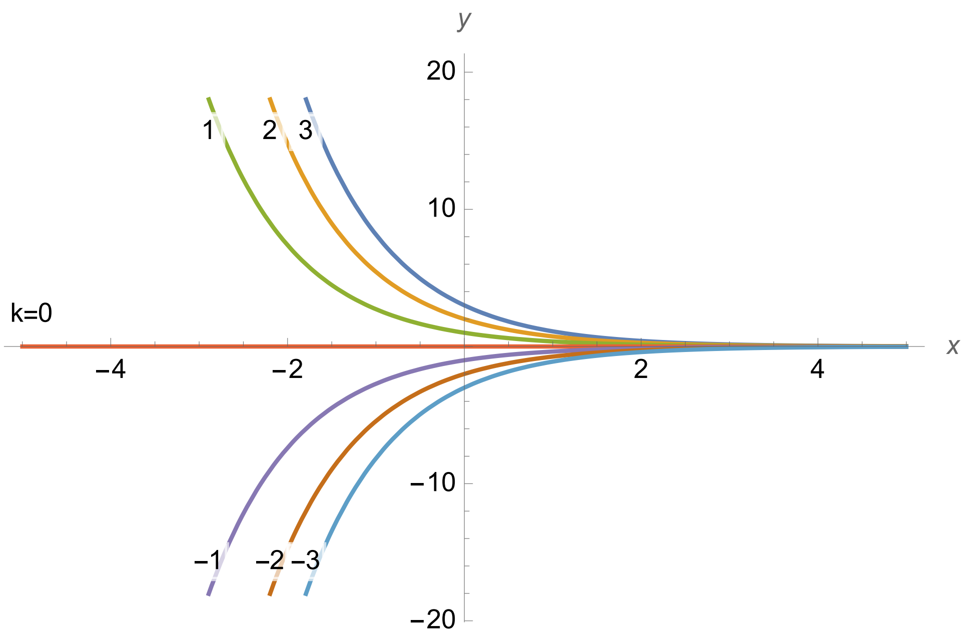
3. $f(x, y) = x^2 - y^2$

   Solution: The level curves are $x^2 - y^2 = k$ for various values of k.



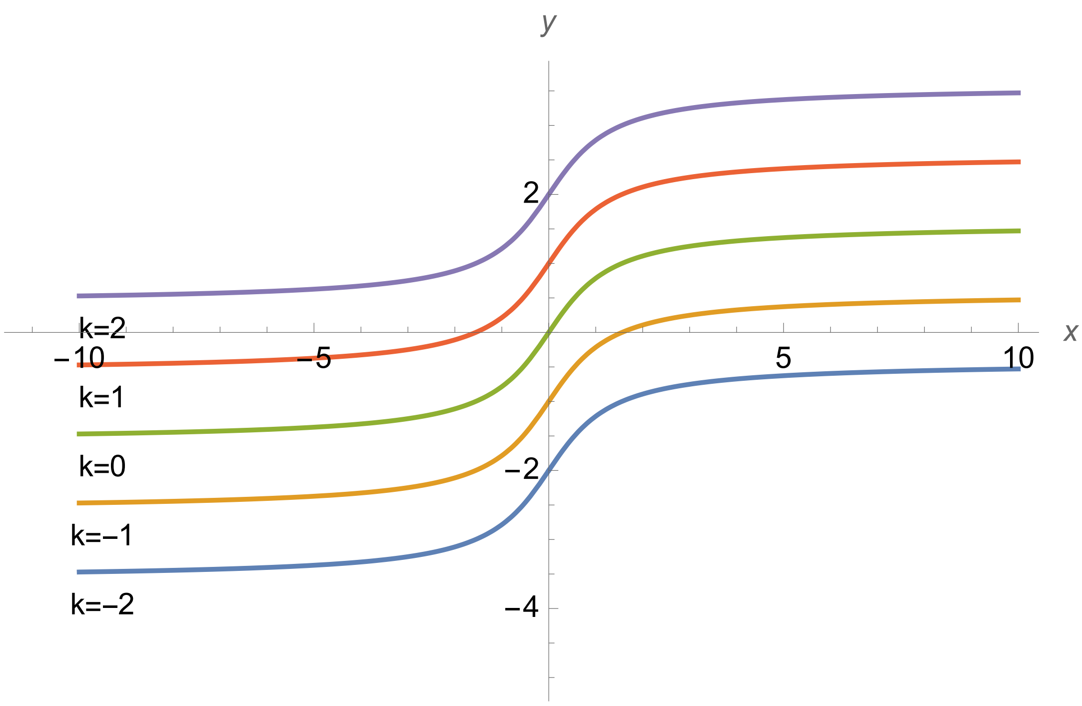
4. $f(x, y) = ye^x$

   Solution: The level curves are $k = ye^x$ or $y = ke^{-x}$.



5. $f(x, y) = y - \tan^{-1}(x)$

   Solution: The level curves are $y = k + \tan^{-1}(x)$ for various values of k.



Cylindrical and Spherical Coordinates

Problem 1: Convert the following Cartesian coordinates into cylindrical and spherical coordinates.

$$(-1,1,1) \quad (-\sqrt{2},\sqrt{2},1) \quad (1,0,\sqrt{3}) \quad (\sqrt{3},-1,2\sqrt{3})$$

Solution: For $(-1,1,1)$:-

$r = \sqrt{x^2+y^2} = \sqrt{(-1)^2+(1)^2} = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{-1}\right) = \frac{3\pi}{4}$

$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{(-1)^2+(1)^2+(1)^2} = \sqrt{3}$

$\phi = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) = \tan^{-1}\left(\frac{\sqrt{2}}{1}\right) = \tan^{-1}(\sqrt{2})$

Thus, the cylinderical coordinates are $(\sqrt{2}, 3\pi/4, 1)$ and spherical coordinates are $\big(\sqrt{3}, 3\pi/4, \tan^{-1}(\sqrt{2})\big)$.

For $(-\sqrt{2},\sqrt{2},1)$:-

$r = \sqrt{x^2+y^2} = \sqrt{(-\sqrt{2})^2+(\sqrt{2})^2} = 2$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{\sqrt{2}}{-\sqrt{2}}\right) = \frac{3\pi}{4}$

$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{(-\sqrt{2})^2+(\sqrt{2})^2+(1)^2} = \sqrt{5}$

$\phi = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) = \tan^{-1}\left(\frac{2}{1}\right) = \tan^{-1}(2)$

Thus, the cylinderical coordinates are $(2, 3\pi/4, 1)$ and spherical coordinates are $\big(\sqrt{5}, 3\pi/4, \tan^{-1}(2)\big)$.

For $(1,0,\sqrt{3})$:-

$r = \sqrt{x^2+y^2} = \sqrt{(1)^2+(0)^2} = 1$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{0}{1}\right) = 0$

$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{(1)^2+(0)^2+(\sqrt{3})^2} = 2$

$\phi = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

Thus, the cylinderical coordinates are $(1, 0, \sqrt{3})$ and spherical coordinates are $\big(2, 0, \pi/6\big)$.

For $(\sqrt{3},-1,2\sqrt{3})$:-

$r = \sqrt{x^2+y^2} = \sqrt{(\sqrt{3})^2+(-1)^2} = 2$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = \frac{11\pi}{6}$

$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{(\sqrt{3})^2+(-1)^2+(2\sqrt{3})^2} = 4$

$\phi = \tan^{-1}\left(\dfrac{\sqrt{x^2+y^2}}{z}\right) = \tan^{-1}\left(\dfrac{2}{2\sqrt{3}}\right) = \pi/6$

Thus, the cylinderical coordinates are $(2, 11\pi/6, 2\sqrt{3})$ and spherical coordinates are $\big(4, 11\pi/6, \pi/6\big)$.

Problem 2: Convert the following cylinderical coordinates into Cartesian and Spherical coordinates.

$$(4, \pi/3, -2) \quad (2, -\pi/2, 1) \quad (\sqrt{2}, 3\pi/4, 2)$$

Solution: For $(4, \pi/3, -2)$:-

$x = r\cos\theta = 4\cos(\pi/3) = 2$

$y = r\sin\theta = 4\sin(\pi/3) = 2\sqrt{3}$

$\rho = \sqrt{r^2+z^2} = \sqrt{(4)^2+(-2)^2} = 2\sqrt{5}$

$\phi = \tan^{-1}\left(\dfrac{r}{z}\right) = \tan^{-1}\left(\dfrac{4}{-2}\right) = \pi - \tan^{-1}(2)$

So, Cartesian coordinates are $(2, 2\sqrt{3}, -2)$ and spherical coordinates are $\big(2\sqrt{5}, \pi/3, \pi - \tan^{-1}(2)\big)$.

For $(2, -\pi/2, 1)$:-

$x = r\cos\theta = 2\cos(-\pi/2) = 0$

$y = r\sin\theta = 2\sin(-\pi/2) = -2$

$\rho = \sqrt{r^2+z^2} = \sqrt{(2)^2+(1)^2} = \sqrt{5}$

$\phi = \tan^{-1}\left(\dfrac{r}{z}\right) = \tan^{-1}\left(\dfrac{2}{1}\right) = \tan^{-1}(2)$

So, Cartesian coordinates are $(0, -2, 1)$ and spherical coordinates are $\big(\sqrt{5}, -\pi/2, \tan^{-1}(2)\big)$.

For $(\sqrt{2}, 3\pi/4, 2)$:-

$x = r\cos\theta = \sqrt{2}\cos(3\pi/4) = -1$

$y = r\sin\theta = \sqrt{2}\sin(3\pi/4) = 1$

$\rho = \sqrt{r^2+z^2} = \sqrt{(\sqrt{2})^2+(2)^2} = \sqrt{6}$

$\phi = \tan^{-1}\left(\dfrac{r}{z}\right) = \tan^{-1}\left(\dfrac{\sqrt{2}}{2}\right) = \tan^{-1}(1/\sqrt{2})$

So, Cartesian coordinates are $(-1, 1, 2)$ and spherical coordinates are $\big(\sqrt{6}, 3\pi/4, \tan^{-1}(1/\sqrt{2})\big)$.

Problem 3: Convert the following spherical coordinates into Cartesian and cylinderical coordinates.

$$(6, \pi/3, \pi/6) \quad (3, \pi/2, 3\pi/4) \quad (2, \pi/2, \pi/2)$$

Solution: For $(6, \pi/3, \pi/6)$:-

$x = \rho \cos\theta \sin\phi = 6\cos(\pi/3)\sin(\pi/6) = 3/2$

$y = \rho \sin\theta \sin\phi = 6\sin(\pi/3)\sin(\pi/6) = 3\sqrt{3}/2$

$z = \rho \cos\phi = 6\cos(\pi/6) = 3\sqrt{3}$

$r = \rho \sin\phi = 6\sin(\pi/6) = 3$

So, cylindrical coordinates are $(3, \pi/3, 3\sqrt{3})$ and Cartesian coordinates are $\big(3/2, 3\sqrt{3}/2, 3\sqrt{3}\big)$.

For $(3, \pi/2, 3\pi/4)$:-

$x = \rho \cos\theta \sin\phi = 3\cos(\pi/2)\sin(3\pi/4) = 0$

$y = \rho \sin\theta \sin\phi = 3\sin(\pi/2)\sin(3\pi/4) = 3/\sqrt{2}$

$z = \rho \cos\phi = 3\cos(3\pi/4) = -3/\sqrt{2}$

$r = \rho \sin\phi = 3\sin(3\pi/4) = 3/\sqrt{2}$

So, cylindrical coordinates are $(3/\sqrt{2}, \pi/2, -3/\sqrt{2})$ and Cartesian coordinates are $\big(0, 3/\sqrt{2}, -3/\sqrt{2}\big)$.

For $(2, \pi/2, \pi/2)$:-

$x = \rho \cos\theta \sin\phi = 2\cos(\pi/2)\sin(\pi/2) = 0$

$y = \rho \sin\theta \sin\phi = 2\sin(\pi/2)\sin(\pi/2) = 2$

$z = \rho \cos\phi = 2\cos(\pi/2) = 0$

$r = \rho \sin\phi = 2\sin(\pi/2) = 2$

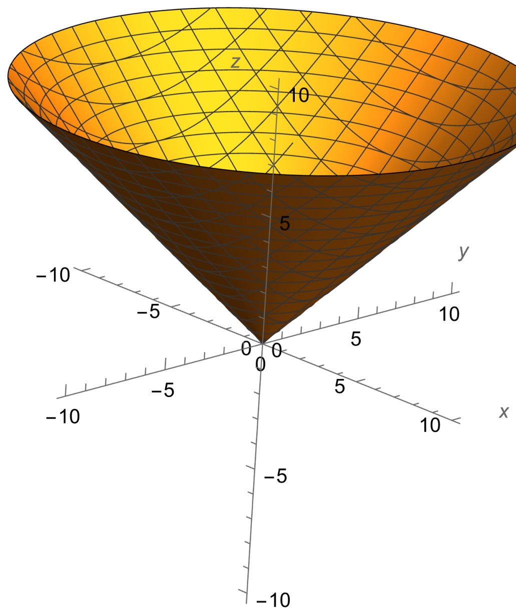
So, cylindrical coordinates are $(2, \pi/2, 0)$ and Cartesian coordinates are $(0, 2, 0)$.

Problem 4: Describe and sketch the surface whose equation in cylinderical coordinates is the following

1. $r = z$

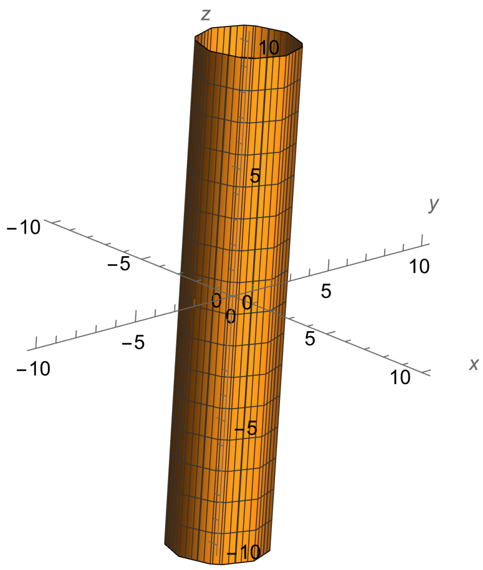
   Solution: Equation in Cartesian coordinates is: $\sqrt{x^2 + y^2} = z \Rightarrow x^2 + y^2 - z^2$ and $z > 0$.

   This represents the part lying above xy-plane, of a cone with vertex $(0, 0, 0)$ and axis being z-axis.



2. $r = 2$

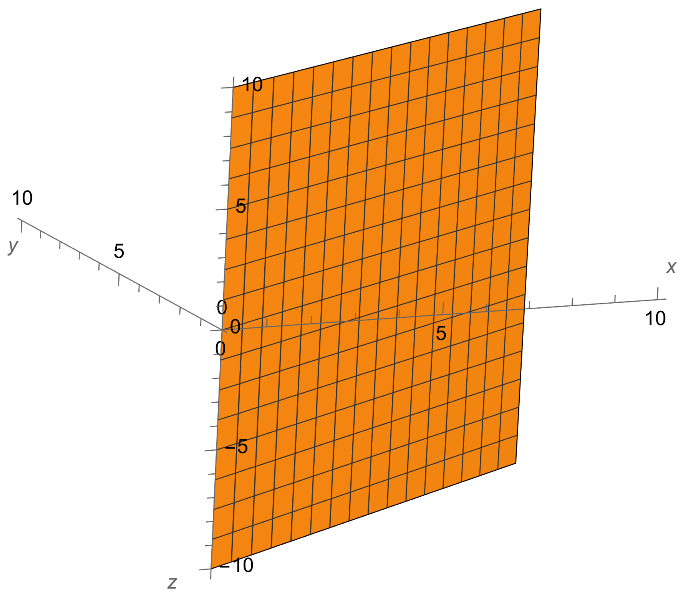
   Solution: Equation in Cartesian coordinates is: $\sqrt{x^2 + y^2} = 2 \Rightarrow x^2 + y^2 = 4$ which is circular cylinder whose axis is the z-axis.



3. $\theta = \pi/6$

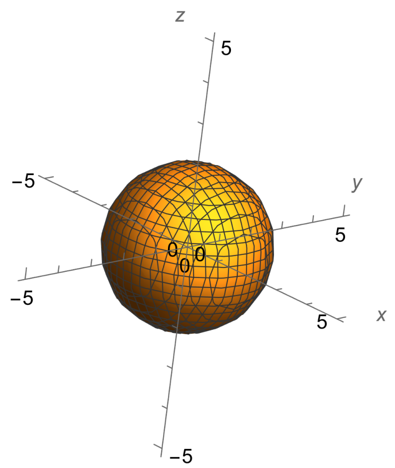
   Solution: Equation in Cartesian coordinates is: $\tan^{-1}\left(\dfrac{y}{x}\right) = \dfrac{\pi}{6}$

   $\Rightarrow \dfrac{y}{x} = \tan(\pi/6)$ and $x > 0,\ y > 0 \Rightarrow x - \sqrt{3}y = 0,\ x > 0,\ y > 0$ which is a half-plane.



4. $r^2 + z^2 = 4$

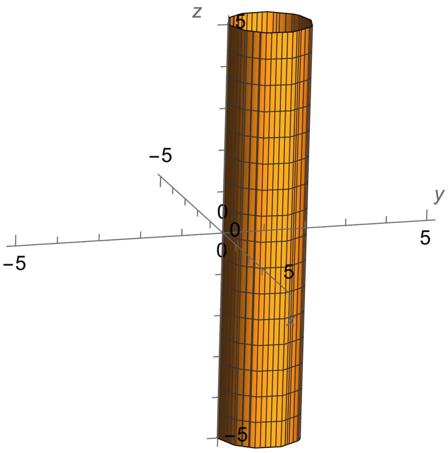
   Solution: Equation in Cartesian coordinates is: $x^2 + y^2 + z^2 = 4$ which is sphere of radius 2 centered at $(0, 0, 0)$.



5. $r = 2\sin\theta$

   Solution: Equation in Cartesian coordinates is: $\sqrt{x^2 + y^2} = 2\dfrac{y}{\sqrt{x^2 + y^2}}$

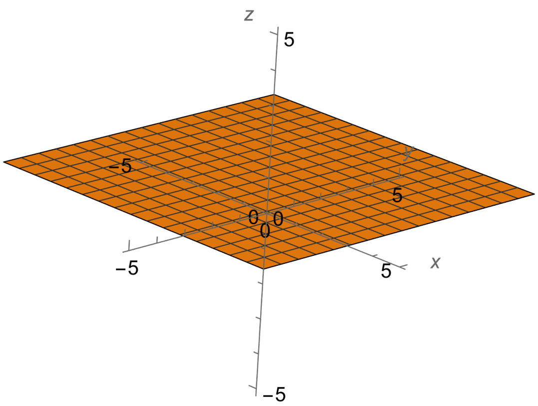
$\Rightarrow x^2 + y^2 - 2y = 0 \Rightarrow x^2 + (y-1)^2 = 1$ which is a circular cylinder with axis passing through $(0, 1, 0)$ and parallel to z-axis.



Problem 5: Describe and sketch the surface whose equation in spherical coordinates is the following

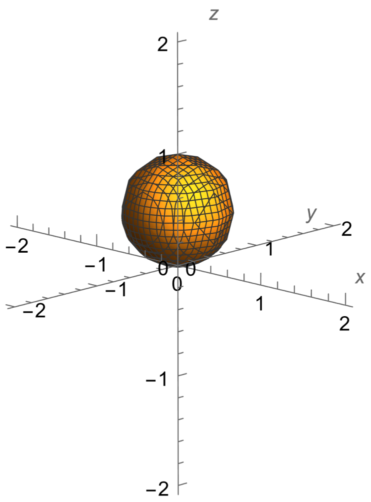
1. $\rho \cos \phi = 1$

   Solution: Equation in Cartesian coordinates is: $z = 1$ which is a plane whose normal vector is $\hat{k}$.



2. $\rho = \cos \phi$

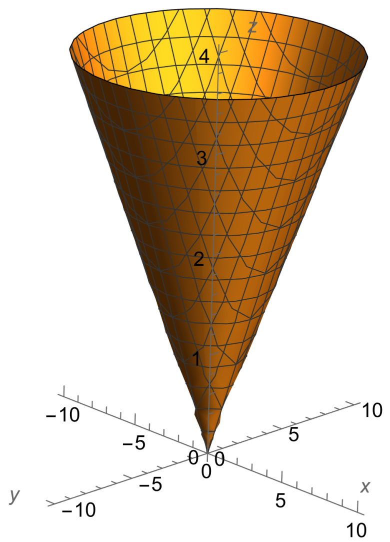
   Solution: Equation in Cartesian coordinates is: $\rho = \dfrac{z}{\rho} \Rightarrow \rho^2 = z \Rightarrow x^2 + y^2 + z^2 = z \Rightarrow$ $x^2 + y^2 + \left(z - \dfrac{1}{2}\right)^2 = \dfrac{1}{4}$ which is a sphere of radius $1/2$ centered at $(0, 0, 1/2)$.



3. $\phi = \pi/3$

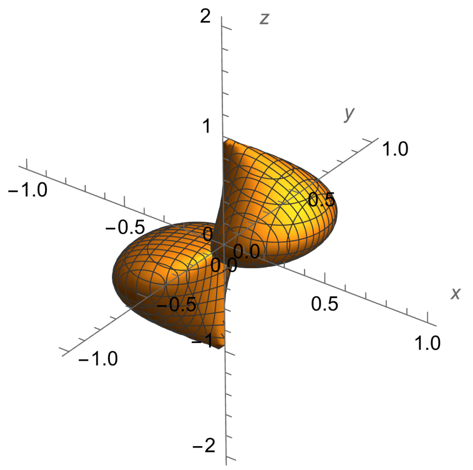
   Solution: $\dfrac{z}{\sqrt{x^2+y^2+z^2}} = \cos(\pi/3) = \dfrac{1}{2} \Rightarrow x^2+y^2+z^2 = 4z^2$ and $z > 0$

   $\Rightarrow x^2+y^2 = 3z^2$ and $z > 0$ which is the upper half (part above the xy-plane) of a cone with vertex at $(0,0,0)$ and axis being the z-axis.



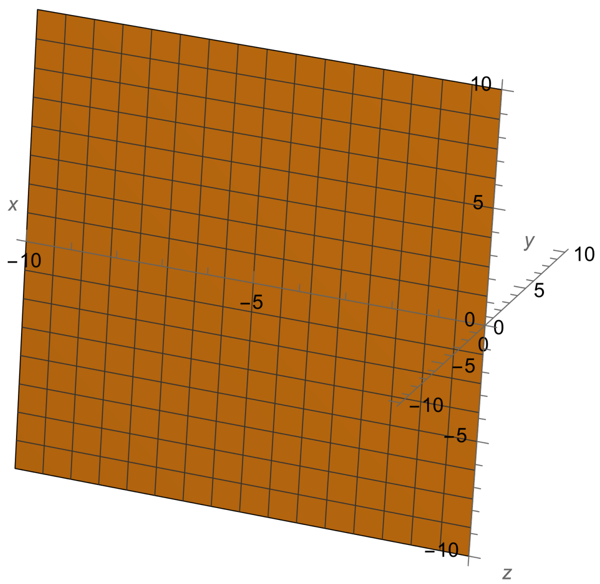
4. $\rho = \cos\theta\cos\phi$

   Solution: $\sqrt{x^2+y^2+z^2} = \dfrac{x}{\sqrt{x^2+y^2}} \dfrac{z}{\sqrt{x^2+y^2+z^2}} \Rightarrow (x^2+y^2+z^2)\left(\sqrt{x^2+y^2}\right) = xz$ which is not a quadric surface. The surface is shown in the figure below:-



5. $\theta = \pi$

   Solution: In Cartesian coordinates we have $\tan^{-1}\left(\dfrac{y}{x}\right) = \pi \Rightarrow y = 0$ and $x < 0$ which is a half-plane.



Problem 6: Write following Cartesian equations in cylinderical and spherical coordinates.

1. $x^2 - x + y^2 + z^2 = 1$

   Solution: In cylindrical coordinates we have:-

   $(r\cos\theta)^2 - (r\cos\theta) + (r\sin\theta)^2 + z^2 = 1 \Rightarrow r^2 - r\cos\theta + z^2 = 1$

   In spherical coordinates we have:-

   $(x^2 + y^2 + z^2) - x = 1 \Rightarrow \rho^2 - \rho\cos\theta\sin\phi = 1$

2. $z = x^2 - y^2$

   Solution: In cylindrical coordinates we have:-

   $z = (r\cos\theta)^2 - (r\sin\theta)^2 \Rightarrow z = r^2\cos 2\theta.$

   In spherical coordinates we have:-

   $\rho\cos\phi = (\rho\cos\theta\sin\phi)^2 - (\rho\sin\theta\sin\phi)^2 \Rightarrow \cos\phi = \rho\sin^2\phi\cos 2\theta.$

3. $z = x^2 + y^2$

   Solution: In cylindrical coordinates we have:-

   $z = (r\cos\theta)^2 + (r\sin\theta)^2 \Rightarrow z = r^2.$

   In spherical coordinates we have:-

   $\rho\cos\phi = (\rho\cos\theta\sin\phi)^2 + (\rho\sin\theta\sin\phi)^2 \Rightarrow \cos\phi = \rho\sin^2\phi.$

4. $x^2 - y^2 - z^2 = 1$

   Solution: In cylindrical coordinates we have:-

   $(r\cos\theta)^2 - (r\sin\theta)^2 - z^2 = 1 \Rightarrow r^2\cos 2\theta - z^2 = 1.$

   In spherical coordinates we have:-

   $(\rho\cos\theta\sin\phi)^2 - (\rho\sin\theta\sin\phi)^2 - (\rho\cos\phi)^2 = 1 \Rightarrow \rho^2\sin^2\phi\cos 2\theta - \rho^2\cos^2\phi = 1.$

Parametric Surfaces

Problem 1: Identify and Sketch the surfaces with the following parametric/vector equations.

1. $\vec{r}\,(u,v) = (u+v)\,\hat{i} + (3-v)\,\hat{j} + (1+4u+5v)\,\hat{k}$

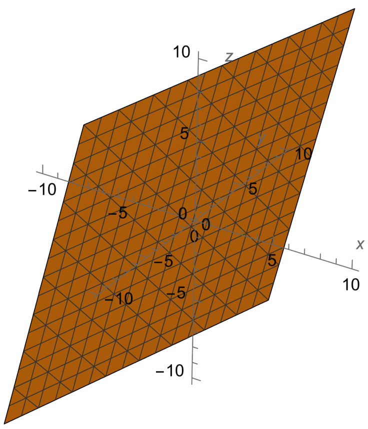
   Solution: In parametric form we have:-

   $x(u,v) = u+v$, $y(u,v) = 3-v$, $z(u,v) = 1+4u+5v$

   Eliminate the parameters u and v to get the Cartesian equation:-

   $$4x - y - z + 4 = 0$$

   which is the equation of a plane as shown in the figure below:-

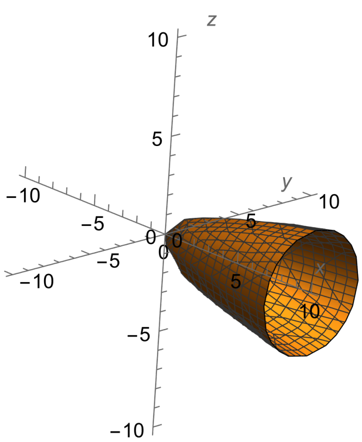


2. $x = u^2\,,\; y = u\cos v\,,\; z = u\sin v$

   Solution: Eliminate the parameters u and v to get the Cartesian equation:-

   $y^2 + z^2 = (u\cos v)^2 + (u\sin v)^2 = u^2(\cos^2 v + \sin^2 v) = u^2 = x$

   Thus, the Cartesian equation of the given surface is $x = y^2 + z^2$ which an elliptic paraboloid with axis being the x-axis and vertex at $(0,0,0)$.

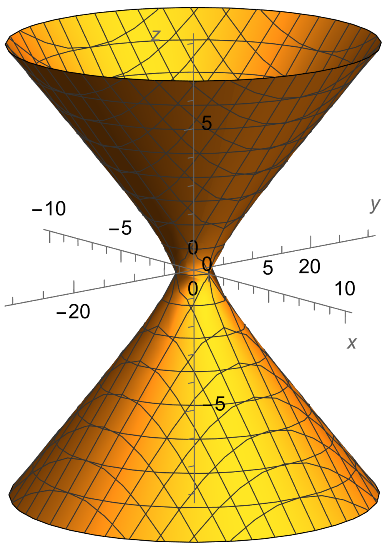


3. $x = (\cos t)(\sec s)\,,\ y = 3\,(\sin t)(\sec s)\,,\ z = \tan s$

   Solution: Eliminate the parameters s and t to get the Cartesian equation:-

   $x^2 + (y/3)^2 = \sec^2 s\,(\cos^2 t + \sin^2 t) = \sec^2 s \Rightarrow x^2 + (y/3)^2 - z^2 = \sec^2 s - \tan^2 s = 1$

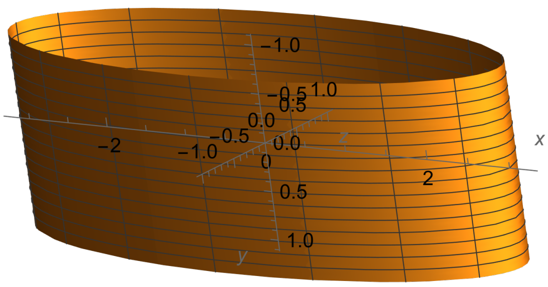
   Thus, the Cartesian equation of the given surface is $x^2 + \dfrac{y^2}{9} - z^2 = 1$ which is a hyperboloid of one sheet whose axis is the z-axis.



4. $x = 3\ \cos t\,,\ y = s\,,\ z = \sin t\,,\ -1 \leq s \leq 1.$

   Solution: Eliminate the parameters s and t to get the Cartesian equation:-

   $(x/3)^2 + y^2 = \cos^2 t + \sin^2 t = 1 \Rightarrow \dfrac{x^2}{9} + y^2 = 1$ which is the equation of an elliptic cylinder whose axis is the z-axis. Since the parameter s has range limited to $-1 \leq s \leq 1$ and $z = s$, we have $-1 \leq z \leq 1$. Thus, the cylinder has finite height.



Problem 2: Find the parametric equation for the following surfaces

1. The plane through the origin that contains the vectors $\hat{\imath} - \hat{\jmath}$ and $\hat{\jmath} - \hat{k}$.

   Solution: Vector equation of a plane passing through a point (x_0, y_0, z_0) and containing two non-parallel vectors $\vec{a}$ and $\vec{b}$ is given by:-

   $$\vec{r}(u,v) = x_0\,\hat{\imath} + y_0\,\hat{\jmath} + z_0\,\hat{k} + u\,\vec{a} + v\,\vec{b}$$

   where u, v range over all real numbers.

   For this case the point is $(0,0,0)$, $\vec{a} = \hat{\imath} - \hat{\jmath}$ and $\vec{b} = \hat{\jmath} - \hat{k}$. Thus, we have

   $$\vec{r}(u,v) = u\,(\hat{\imath} - \hat{\jmath}) + v\,(\hat{\jmath} - \hat{k}) = u\,\hat{\imath} + (v-u)\,\hat{\jmath} - v\,\hat{k}$$

   Therefore, the parametric equation of the given plane is given by:-

   $$\boxed{x = u\,, \quad y = v - u\,, \quad z = -v}$$

2. The part of the hyperboloid $4x^2 - 4y^2 - z^2 = 4$ that lies in front of the yz-plane.

   Solution: The given equation is $\dfrac{x^2}{1} - \dfrac{y^2}{1} - \dfrac{z^2}{4} = 1$ or $\dfrac{x^2}{1} - \left(\dfrac{y^2}{1} + \dfrac{z^2}{4}\right) = 1$

   Using $\sec^2\phi - \tan^2\phi = 1$ we let $x = \sec\phi$ and $\dfrac{y^2}{1} + \dfrac{z^2}{4} = \tan^2\phi$.

   Using $\sin^2\theta + \cos^2\theta = 1$ we let $y = \cos\theta\tan\phi$ and $\dfrac{z}{2} = \sin\theta\tan\phi$.

   Since we want only the part that lies in front of the yz-plane, we have to ensure $x > 0$.

   So, we limit the range of ϕ to $-\pi/2 < \phi < \pi/2$.

   Therefore, the parametric equation of the given surface is:-

   $$\boxed{x = \sec\phi\,, \quad y = \cos\theta\tan\phi\,, \quad z = 2\sin\theta\tan\phi\,, \quad -\pi/2 < \phi < \pi/2\,,\ \theta \in \mathbb{R}}$$

   ALTERNATIVELY,

   Let $y = u$, $z = v$. Then $4x^2 = 4 + 4u^2 + v^2 \Rightarrow x^2 = \dfrac{1}{4}(4 + 4u^2 + v^2)$.

   $\Rightarrow x = \pm\dfrac{1}{2}\sqrt{4 + 4u^2 + v^2}$. Since we want the part which lies in front of yz-plane, we have to choose $x > 0$. Therefore, the parametric equation of the given surface is:-

   $$\boxed{x = \frac{1}{2}\sqrt{4 + 4u^2 + v^2}\,, \quad y = u\,, \quad z = v\,, \quad u \in \mathbb{R}\,,\ v \in \mathbb{R}}$$

3. The part of the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$ that lies to the left of the xz-plane.

   Solution: To left of xz-plane, we have $y < 0$. Let $x = u$, $z = v$. Then

   $2y^2 = 1 - u^2 - 3v^2 \Rightarrow y^2 = \dfrac{1}{2}(1 - u^2 - 3v^2) \Rightarrow y = \pm\sqrt{0.5(1 - u^2 - 3v^2)}$

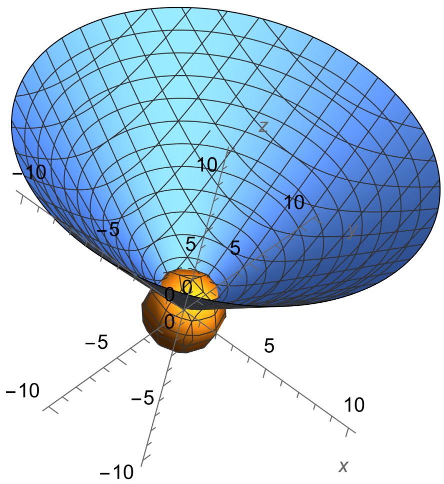
   We choose $y < 0$ since we want the part to the left of xz-plane.

Therefore, the parametric equation of the given surface is:-

$$\boxed{x = u\,, \quad y = \sqrt{0.5(1-u^2-3v^2)}\,, \quad z = v\,, \quad u, v \in \mathbb{R} \text{ such that } u^2 + 3v^2 \leq 1}$$

4. The part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = \sqrt{x^2 + y^2}$.

Solution: Shown below are both the given sphere and the given cone.



We want the part of sphere lying above the cone. So we want to have $z > \sqrt{x^2 + y^2}$

This implies $z^2 > x^2 + y^2$.

But since we are considering points lying on the sphere we have $z^2 = 4 - (x^2 + y^2)$.

Thus, we should have $4 - (x^2 + y^2) > x^2 + y^2 \Rightarrow 2(x^2 + y^2) < 4 \Rightarrow x^2 + y^2 < 2$.

Now let $x = u$, $y = v$. Then $z^2 = 4 - u^2 - v^2 \Rightarrow z = \pm\sqrt{4 - u^2 - v^2}$.

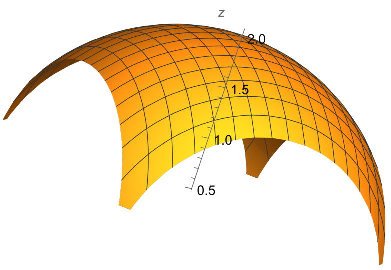
Since we want the part of sphere lying above the cone, we have to choose $z > 0$.

Thus, $z = \sqrt{4 - u^2 - v^2}$ and $u^2 + v^2 < 2$.

Therefore, the parametric equation of the given surface is:-

$$\boxed{x = u\,, \quad y = v\,, \quad z = \sqrt{4 - u^2 - v^2}\,, \quad u, v \in \mathbb{R} \text{ such that } u^2 + v^2 < 2}$$

The surface looks as below:-



5. The part of the cylinder $x^2+z^2=9$ that lies above the xy-plane and between the planes $y=-4$ and $y=4$.

   Solution: The given cylinder has its axis to be y-axis. Let $x=u$ and $y=v$.

   Since we only want the part that lies between the planes $y=-4$ and $y=4$, we have $-4<v<4$.

   Now, $z^2=9-u^2 \Rightarrow z=\pm\sqrt{9-u^2}$. But we choose $z>0$ because we are considering the part of cylinder that lies above the xy-plane.

   Therefore, the parametric equation of the given surface is:-

$$\boxed{x=u\,,\quad y=v\,,\quad z=\sqrt{9-u^2}\,,\quad -3\le u\le 3\,,\ -4<v<4}$$

   The given surface looks as shown below:-