Problem 1: Find the vertex, focus and directrix of the following parabolas and sketch its graph.

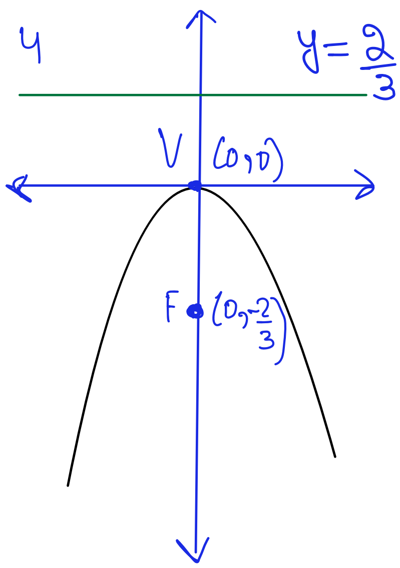
1. $3x^2 + 8y = 0$.
2. $y^2 + 6y + 2x + 1 = 0$.
3. $2x^2 - 16x - 3y + 38 = 0$.

① $3x^2 + 8y = 0 \Rightarrow \underbrace{x^2 = -\frac{8y}{3}}_{\text{Standard Form 4}} \Rightarrow 4P = \frac{8}{3} \Rightarrow P = \frac{2}{3}$

$\Rightarrow$ Focus is $\boxed{\left(0, -\frac{2}{3}\right)}$,

directrix is $\boxed{y = \frac{2}{3}}$

and vertex is $\boxed{(0,0)}$.



② $y^2 + 6y + 2x + 1 = 0$

$\Rightarrow \underbrace{y^2 + 6y + 3^2} - 3^2 + 2x + 1 = 0 \Rightarrow (y+3)^2 - 9 + 2x + 1 = 0$

$\Rightarrow (y+3)^2 = -2x + 8 \Rightarrow (y+3)^2 = -2(x-4)$

Let $\left.\begin{array}{l} Y = y+3 \\ X = x-4 \end{array}\right\} \Rightarrow Y^2 = -2X \longrightarrow$ Standard Form 2 in new coordinates

$\Rightarrow 4P = 2 \Rightarrow P = \frac{1}{2} \Rightarrow$ Focus is $\left.\begin{array}{l} X = -\frac{1}{2} \\ Y = 0 \end{array}\right\} \Rightarrow \begin{array}{l} x-4 = \frac{-1}{2} \\ y+3 = 0 \end{array}$

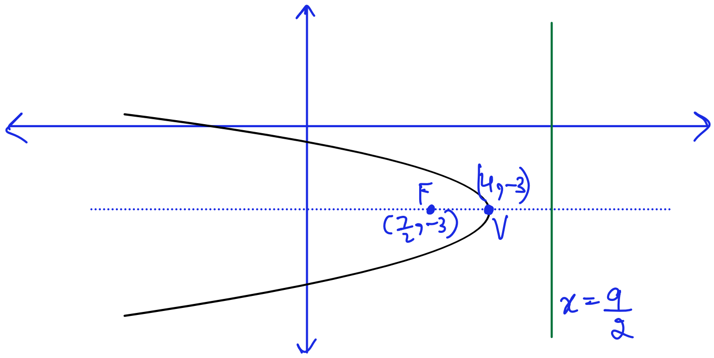
$\Downarrow$

Directrix is $\qquad \Rightarrow$ Focus is $\boxed{\left(\frac{7}{2}, -3\right)}$

$X = \frac{1}{2}$

$\Rightarrow x - 4 = \frac{1}{2} \Rightarrow \boxed{x = \frac{9}{2}}$

Vertex is at $X = 0,\ Y = 0 \Rightarrow x-4 = 0,\ y+3 = 0 \Rightarrow \boxed{(4, -3)}$



③ $2x^2-16x-3y+38=0 \Rightarrow 2(x^2-8x)-3y+38=0$

$\Rightarrow 2(\underbrace{x^2-8x+4^2}-4^2)-3y+38=0 \Rightarrow 2(x+4)^2-32-3y+38=0$

$\Rightarrow 2(x-4)^2=3y+32-38 \Rightarrow 2(x+4)^2=3y-6$

$\Rightarrow (x-4)^2=\frac{3}{2}(y-2)$

Let $\left.\begin{array}{l} X=x-4 \\ Y=y-2 \end{array}\right\} \Rightarrow X^2=\frac{3}{2}Y \Rightarrow 4p=\frac{3}{2} \Rightarrow p=\frac{3}{8}$

(Standard form 3)

$\Rightarrow$ Focus in new coordinates is at $X=0, Y=\frac{3}{8}$

and directrix in new coordinates is $Y=-\frac{3}{8}$

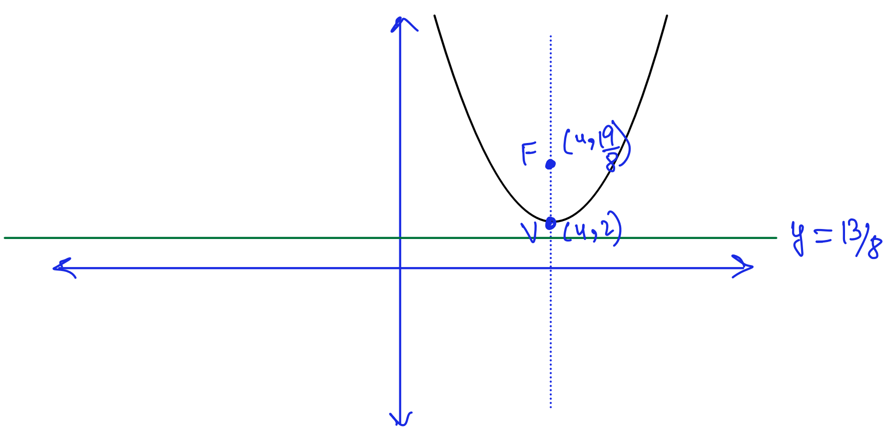
$\Rightarrow$ Focus is at $x-4=0, y-2=\frac{3}{8} \Rightarrow \left(4, 2+\frac{3}{8}\right) = \boxed{\left(4, \frac{19}{8}\right)}$

and equation of directrix is $y-2=-\frac{3}{8} \Rightarrow y=2-\frac{3}{8}$

$\Rightarrow \boxed{y=\frac{13}{8}}$

Vertex is at $X=0, Y=0$

$\Rightarrow x-4=0, y-2=0 \Rightarrow \boxed{(4,2)}$



Problem 2: Find the vertices and foci of the following ellipses and sketch its graph.

1. $x^2 + 9y^2 = 9$.
2. $9x^2 - 18x + 4y^2 = 27$.
3. $x^2 + 3y^2 + 2x - 12y + 10 = 0$.

① $x^2+9y^2=9 \Rightarrow \frac{x^2}{9}+\frac{y^2}{1}=1 \Rightarrow a^2=9,\ b^2=1 \Rightarrow a=3,\ b=1$

(Standard Form 1)

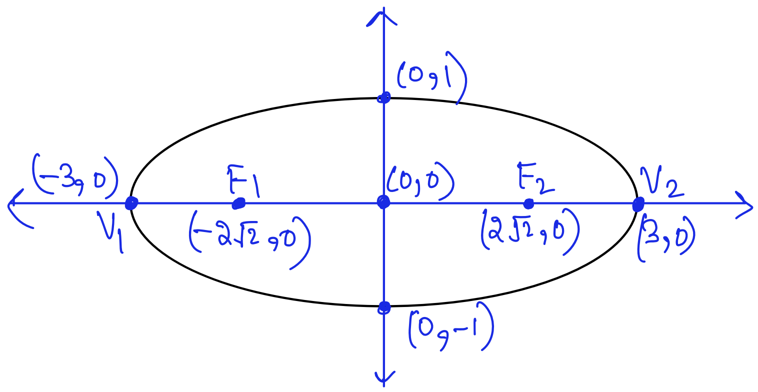
$\Downarrow$

$c^2=a^2-b^2=9-1=8$

$\Rightarrow$ Vertices are $(\pm a, 0) = \boxed{(\pm 3, 0)}$

$\Rightarrow$ Foci are $(\pm c, 0) = (\pm\sqrt{8}, 0)$

$= \boxed{(\pm 2\sqrt{2}, 0)}$



② $9x^2-18x+4y^2=27 \Rightarrow 9(x^2-2x)+4y^2=27$

$\Rightarrow 9(x^2-2x+1-1)+4y^2=27 \Rightarrow 9(x-1)^2-9+4y^2=27$

$\Rightarrow 9(x-1)^2+4y^2=27+9=36 \Rightarrow \frac{(x-1)^2}{4}+\frac{y^2}{9}=1$

Let $\left.\begin{array}{l} X=x-1 \\ Y=y \end{array}\right\} \Rightarrow \frac{X^2}{4}+\frac{Y^2}{9}=1$

$\hookrightarrow$ Standard Form 2

$\Rightarrow b^2=4,\ a^2=9 \Rightarrow b=2,\ a=3$

For standard form 2 :-

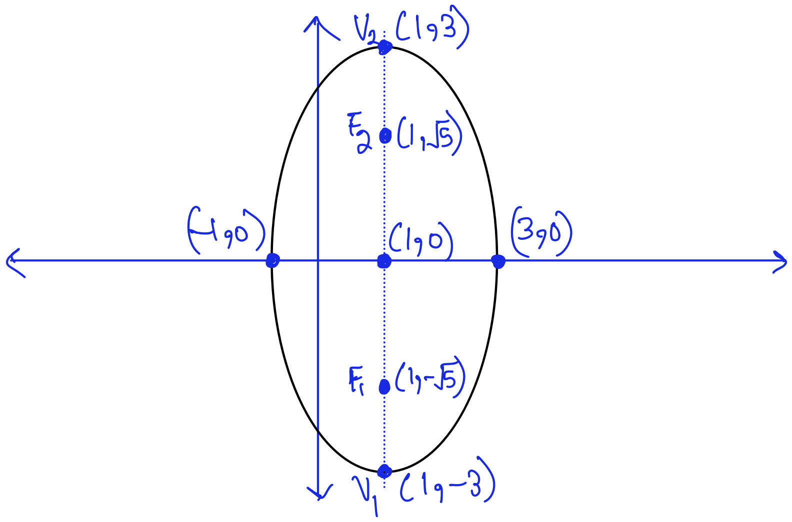
$\hookrightarrow$ Vertices are $X=0,\ Y=\pm 3 \Rightarrow x-1=0,\ y=\pm 3$

$\Rightarrow \boxed{(1, \pm 3)}$

$c^2 = a^2 - b^2 = 9-4 = 5 \Rightarrow c = \sqrt{5}$

$\hookrightarrow$ Foci are $X=0, Y=\pm\sqrt{5} \Rightarrow x-1=0, y=\pm\sqrt{5}$

$\Rightarrow \boxed{(1, \pm\sqrt{5})}$



③ $x^2+3y^2+2x-12y+10=0 \Rightarrow x^2+2x+1-1+3(y^2-4y+4-4)+10=0$

$\Rightarrow (x+1)^2-1+3(y-2)^2-12+10=0$

$\Rightarrow (x+1)^2+3(y-2)^2=3 \Rightarrow \frac{(x+1)^2}{3}+\frac{(y-2)^2}{1}=1$

Let $\left.\begin{array}{l} X = x+1 \\ Y = y-2 \end{array}\right\} \Rightarrow \frac{X^2}{3}+\frac{Y^2}{1}=1$ $\hookrightarrow$ Standard Form 1

$a^2=3,\ b^2=1 \Rightarrow a=\sqrt{3},\ b=1$

$\Downarrow$

$c^2 = 3-1 = 2 \Rightarrow c=\sqrt{2}$

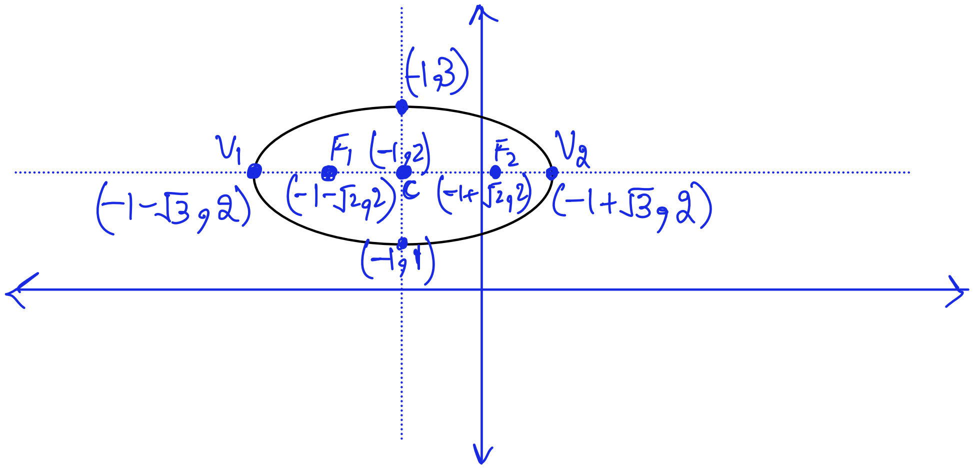
For Standard form 1,

$\hookrightarrow$ the vertices are $X=\pm\sqrt{3},\ Y=0 \Rightarrow x+1=\pm\sqrt{3},\ y-2=0$

$\Rightarrow \boxed{(-1\pm\sqrt{3}, 2)}$

$\hookrightarrow$ the foci are $X=\pm\sqrt{2},\ Y=0$

$\Rightarrow x+1=\pm\sqrt{2},\ y-2=0 \Rightarrow \boxed{(-1\pm\sqrt{2}, 2)}$



Problem 3: Find the vertices, foci and asymptotes of the following hyperbolas and sketch its graph.

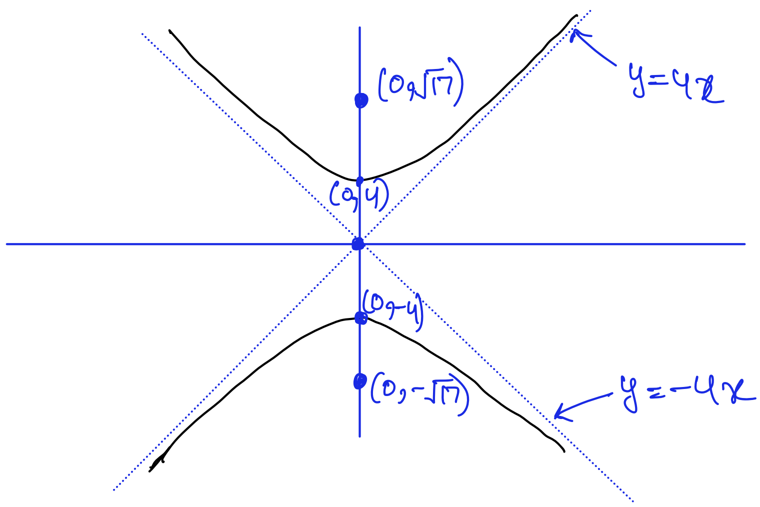
1. $y^2 - 16x^2 = 16$
2. $x^2 - y^2 + 2y = 2$
3. $9y^2 - 4x^2 - 36y - 8x = 4$.

(1) $y^2 - 16x^2 = 16 \Rightarrow \frac{y^2}{16} - \frac{x^2}{1} = 1$ (Standard form 2)

$\underbrace{a^2 = 16,\ b^2 = 1}_{c^2 = a^2 + b^2 = 17 \Rightarrow c = \sqrt{17}} \Rightarrow a = 4,\ b = 1$ $\Rightarrow$ vertices are $\boxed{(0, \pm 4)}$, Foci are $\boxed{(0, \pm\sqrt{17})}$

For the asymptotes, put the constant term to 0.

$\Rightarrow y^2 - 16x^2 = 0 \Rightarrow (y-4x)(y+4x) = 0 \Rightarrow \boxed{y = \pm 4x}$



② $x^2-y^2+2y=2 \Rightarrow x^2-(y^2-2y)=2 \Rightarrow x^2-(\underbrace{y^2-2y+1}_{(y-1)^2}-1)=2$

$\Rightarrow x^2-(y-1)^2+1=2 \Rightarrow x^2-(y-1)^2=1$

Let $\left.\begin{array}{l} X=x \\ Y=y-1 \end{array}\right\} \Longrightarrow X^2-Y^2=1$ (Standard form 1)

$a^2=1, b^2=1 \Rightarrow a=1, b=1$

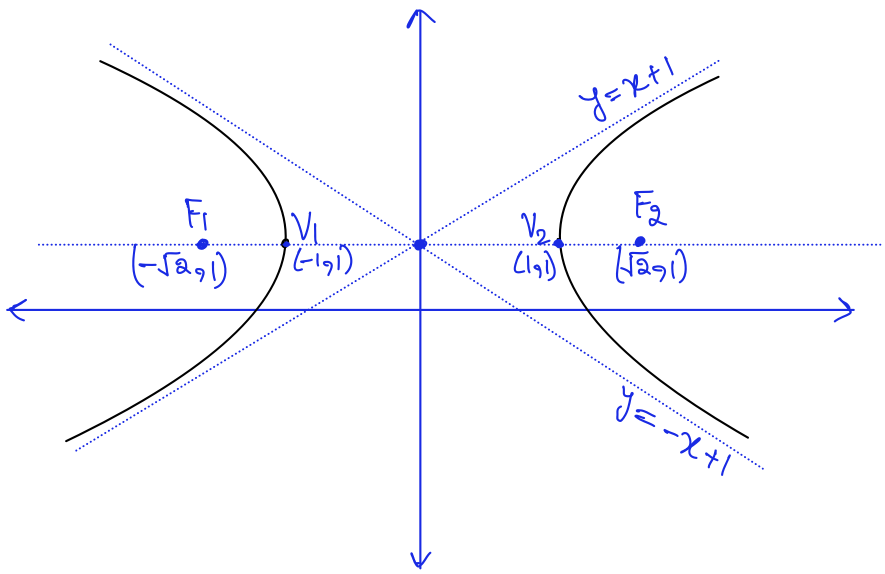
$c^2=a^2+b^2=2 \Rightarrow c=\sqrt{2}$

$\Rightarrow$ Vertices are at $X=\pm1, Y=0 \Rightarrow x=\pm1, y-1=0 \Rightarrow \boxed{(\pm1,1)}$

$\Rightarrow$ Foci are at $X=\pm\sqrt{2}, Y=0 \Rightarrow x=\pm\sqrt{2}, y-1=0 \Rightarrow \boxed{(\pm\sqrt{2},1)}$

To find asymptotes, put the constant term to 0,

$\Rightarrow X^2-Y^2=0 \Rightarrow Y=\pm X \Rightarrow y-1=\pm x \Rightarrow \boxed{\begin{array}{l} y=x+1 \\ y=-x+1 \end{array}}$



③ $9y^2-4x^2-36y-8x=4 \Rightarrow 9(y^2-4y)-4(x^2+2x)=4$

$\Rightarrow 9(\underbrace{y^2-4y+4}_{(y-2)^2}-4)-4(\underbrace{x^2+2x+1}_{(x+1)^2}-1)=4$

$\Rightarrow 9(y-2)^2-36-4(x+1)^2+4=4 \Rightarrow 9(y-2)^2-4(x+1)^2=36$

$\Rightarrow \dfrac{(y-2)^2}{4}-\dfrac{(x+1)^2}{9}=1$

Let $\left.Y=y-2, X=x+1\right\} \Rightarrow Y^2/4 - X^2/9 = 1$ (Standard form 2)

$\Rightarrow \quad a^2 = 4,\ b^2 = 9 \quad \Rightarrow a = 2 \quad \text{and} \quad c^2 = 4+9 = 13 \Rightarrow c = \sqrt{13}$

Vertices are at $X = 0,\ Y = \pm 2 \Rightarrow x+1 = 0,\ y-2 = \pm 2$

$\Rightarrow$ Vertices at $(-1, 2\pm 2) = \boxed{(-1,0) \text{ and } (-1,4)}$

Foci are at $X = 0,\ Y = \pm\sqrt{3} \Rightarrow x+1 = 0,\ y-2 = \pm\sqrt{3}$

$\Rightarrow$ Foci at $\boxed{(-1, 2\pm\sqrt{13})}$

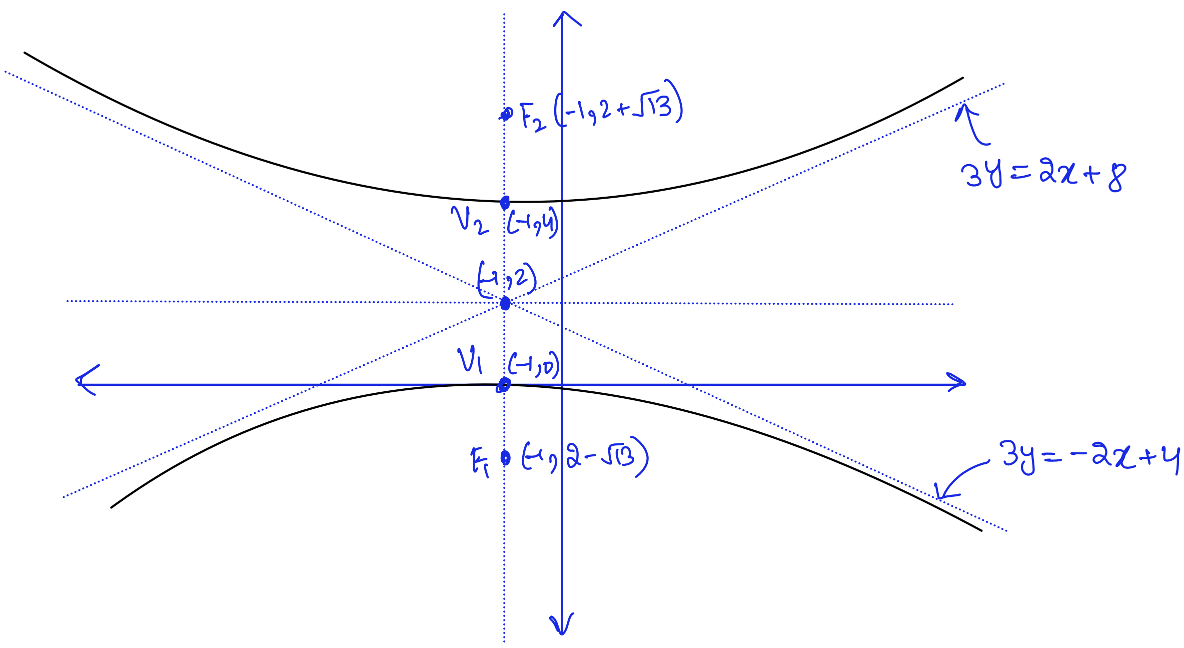
For asymptotes put constant term in $\frac{Y^2}{4} - \frac{X^2}{9} = 1$ to 0.

$$\Rightarrow \frac{Y^2}{4} - \frac{X^2}{9} = 0 \Rightarrow \frac{Y}{2} = \pm\frac{X}{3} \Rightarrow \frac{y-2}{2} = \pm\frac{x+1}{3}$$

$$\Rightarrow y-2 = \pm\frac{2}{3}(x+1) \Rightarrow y = \frac{2}{3}x + \frac{2}{3} + 2 \text{ and } y = \frac{-2}{3}x - \frac{2}{3} + 2$$

$$\Rightarrow y = \frac{2}{3}x + \frac{8}{3} \quad \text{and} \quad y = -\frac{2}{3}x + \frac{4}{3}$$

$$\Rightarrow \boxed{3y = 2x+8 \quad \text{and} \quad 3y = -2x+4}$$



Problem 4: Identify the type of conic whose equation is given and find the vertices and foci.

1. $x^2 = 4y - 2y^2$.
2. $3x^2 - 6x - 2y = 1$.
3. $x^2 - 2x + 2y^2 - 8y + 7 = 0$.

① $x^2 = 4y - 2y^2 \Rightarrow x^2 = -2(y^2 - 2y) \Rightarrow x^2 = -2(\underbrace{y^2 - 2y + 1}_{(y-1)^2} - 1)$

$\Rightarrow x^2 = -2(y-1)^2 + 2$

$\Rightarrow x^2 + 2(y-1)^2 = 2 \Rightarrow \frac{x^2}{2} + \frac{(y-1)^2}{1} = 1$

Let $\left.\begin{array}{l} X = x \\ Y = y-1 \end{array}\right\} \Rightarrow \frac{X^2}{2} + \frac{Y^2}{1} = 1 \longrightarrow \boxed{\text{Ellipse}}$ in standard form 1

$a^2 = 2, \; b^2 = 1 \Rightarrow c^2 = a^2 - b^2 = 2 - 1 = 1$

$\Rightarrow a = \sqrt{2}, \quad c = 1$

Vertices are at $X = \pm\sqrt{2}, \; Y = 0 \Rightarrow x = \pm\sqrt{2}, \; y - 1 = 0$

$\Rightarrow \boxed{(\pm\sqrt{2}, 1)}$

Foci are at $X = \pm 1, \; Y = 0 \Rightarrow x = \pm 1, \; y - 1 = 0$

$\Rightarrow \boxed{(\pm 1, 1)}$

② $3x^2 - 6x - 2y = 1 \Rightarrow 3(x^2 - 2x) = 2y + 1 \Rightarrow 3(\underbrace{x^2 - 2x + 1}_{(x-1)^2} - 1) = 2y + 1$

$\Rightarrow 3(x-1)^2 - 3 = 2y + 1$

$\Rightarrow 3(x-1)^2 = 2y + 4 \Rightarrow (x-1)^2 = \frac{2}{3}(y+2)$

Let $\left.\begin{array}{l} X = x-1 \\ Y = y+2 \end{array}\right\} \Rightarrow X^2 = \frac{2}{3} Y \longrightarrow \boxed{\text{Parabola}}$ in standard form 3

$\Rightarrow$ Vertex is at $X = 0, \; Y = 0 \Rightarrow x - 1 = 0, \; y + 2 = 0$

$\Rightarrow \boxed{(1, -2)}$

and Focus is at $(0, p)$ (X, Y) where $4p = \frac{2}{3} \Rightarrow p = \frac{1}{6}$

$\Rightarrow$ Focus is at $X=0,\ Y=\frac{1}{6} \Rightarrow x-1=0,\ y+2=\frac{1}{6}$

$\Rightarrow x=1,\ y=-2+\frac{1}{6}=\frac{-11}{6} \Rightarrow \boxed{\left(1,-\frac{11}{6}\right)}$

3. $x^2-2x+2y^2-8y+7=0$

$\Rightarrow \underbrace{x^2-2x+1}_{(x-1)^2}-1+2(\underbrace{y^2-4y+4}_{(y-2)^2}-4)+7=0$

$\Rightarrow (x-1)^2-1+2(y-2)^2-8+7=0$

$\Rightarrow (x-1)^2+2(y-2)^2=2 \Rightarrow \frac{(x-1)^2}{2}+\frac{(y-2)^2}{1}=1$

Let $\left.\begin{array}{l} X=x-1 \\ Y=y-2 \end{array}\right\} \Rightarrow \frac{X^2}{2}+\frac{Y^2}{1}=1 \rightarrow \boxed{\text{Ellipse}}$ in standard form!

$a^2=2,\ b^2=1 \Rightarrow c^2=2-1=1$

$\Rightarrow a=\sqrt{2},\ c=1$

$\Rightarrow$ vertices are at $X=\pm\sqrt{2},\ Y=0 \Rightarrow x-1=\pm\sqrt{2},\ y-2=0$

$\Rightarrow \boxed{(1\pm\sqrt{2},2)}$

$\Rightarrow$ Foci are at $X=\pm c,\ Y=0 \Rightarrow x-1=\pm 1,\ y-2=0$

$\Rightarrow (1\pm 1,2) = \boxed{(0,2) \text{ and } (2,2)}$

Problem 5: Find an equation for the conic that satisfies the following conditions.

1. Parabola with vertex $(2,2)$ and focus $(3,2)$.
2. Ellipse with center $(-1,4)$, vertex $(-1,0)$, focus $(-1,6)$.
3. Hyperbola with foci $(2,0)$, $(2,8)$, asymptotes $y=3+\frac{1}{2}x$ and $y=5-\frac{1}{2}x$.

1. Vertex at $(2,2) \Rightarrow$ we shift the origin at $(2,2)$

Let (X,Y) be new coordinates $\Rightarrow X=x-2$

$Y=y-2$

In new coordinates, focus is at $(3-2, 2-2) = (1, 0)$

$\Rightarrow P = 1$. Since focus is on +ve-X axis, the parabola is in standard form 1

Thus, its equation in new coordinates is :-

$$Y^2 = 4PX \Rightarrow Y^2 = 4X$$

Therefore in old/original coordinates (x, y) we have :-

$$(y-2)^2 = 4(x-2) \Rightarrow y^2 - 4y + 4 = 4x - 8$$

$\Rightarrow \boxed{y^2 - 4y - 4x + 12 = 0}$ is the equation of the given parabola.

(2) Centre of ellipse is at $(-1, 4)$, vertex at $(-1, 0)$, focus at $(-1, 6)$

Thus, we shift the origin to $(-1, 4)$.

Let (X, Y) be new coordinates thus obtained.

$$\Rightarrow X = x+1, \; Y = y-4$$

Standard Form 2

In new coordinates, vertex is at $(\overset{x,y}{(-1,0)} \rightarrow) \; (-1+1, 0-4) = (0, -4)$ (X, Y)

In new coordinates focus is at $(-1+1, 6-4) = (0, 2)$

Thus, $a = 4$ and $c = 2 \Rightarrow b^2 = a^2 - c^2 = 4^2 - 2^2 = 16 - 4 = 12$

$\Rightarrow$ In new coordinates, the ellipse has equation :-

$$\frac{X^2}{b^2} + \frac{Y^2}{a^2} = 1 \equiv \frac{X^2}{12} + \frac{Y^2}{16} = 1 \Rightarrow \frac{(x+1)^2}{12} + \frac{(y-4)^2}{16} = 1 \text{ [required eqn]}$$

can be left as final answer.

$$\Rightarrow 16(x+1)^2 + 12(y-4)^2 = 192$$

$$\Rightarrow 16x^2 + 32x + 16 + 12y^2 - 96y + 192 = 192$$

$\Rightarrow \boxed{16x^2 + 12y^2 + 32x - 96y + 16 = 0}$ is the required equation

③ Hyperbola with foci $(2,0)$, $(2,8)$

The asymptotes are $y = 3 + \frac{1}{2}x$ and $y = 5 - \frac{1}{2}x$

The intersection of asymptotes is the centre

Equating y : $3 + \frac{1}{2}x = 5 - \frac{1}{2}x \Rightarrow x = 5 - 3 = 2 \Rightarrow y = 3 + \frac{2}{2} = 4$

$\Rightarrow$ The centre is at $(2,4)$

So, we shift the origin at $\underline{(2,4)}$

The new coordinates (X, Y) thus obtained are given by :-

$$X = x - 2, \quad Y = y - 4$$

In new coordinates, the foci are $(2-2, 0-4), (2-2, 8-4)$
$= (0, -4), (0, 4)$

$$\Rightarrow c = 4 \text{ ——— } ①$$

In new coordinates the asymptotes are given by :-

$$Y + 4 = 3 + \frac{1}{2}(X + 2) \text{ and } Y + 4 = 5 - \frac{1}{2}(X + 2)$$

$$\Rightarrow Y = \frac{1}{2}X \quad \text{and} \quad Y = -\frac{1}{2}X$$

Now, the foci are on Y-axis and thus the equation of hyperbola is of the form $\frac{Y^2}{a^2} - \frac{X^2}{b^2} = 1$ and asymptotes are of the form $Y = \pm \frac{a}{b}X$

$$\Rightarrow \frac{a}{b} = \frac{1}{2} \Rightarrow b = 2a$$

From ①, $c = 4 \Rightarrow c^2 = 16 \Rightarrow a^2 + b^2 = 16 \Rightarrow a^2 + (2a)^2 = 16$

$$\Rightarrow a^2 = \frac{16}{5} \Rightarrow b^2 = 4a^2 = \frac{64}{5}$$

$\Rightarrow$ equation of hyperbola is $\frac{Y^2}{16/5} - \frac{X^2}{64/5} = 1$

$$\Rightarrow \frac{5Y^2}{16} - \frac{5X^2}{64} = 1 \Rightarrow 20Y^2 - 5X^2 = 64$$

$\Rightarrow 20(y-4)^2 - 5(x-2)^2 = 64 \longrightarrow$ can be left here as final answer.

$\Rightarrow 20y^2 - 160y + 320 - 5x^2 + 20x - 20 = 64$

$\Rightarrow \boxed{20y^2 - 5x^2 - 160y + 20x + 236 = 0}$ is the required equation.