Problem 1: Sketch the following parametric curves.

1. $x = 2t + 1$, $y = t^2 + 1$, $t \in \mathbb{R}$.
2. $x = 1 + \sin\theta$, $y = -1 + 2\cos\theta$, $0 \le \theta \le 2\pi$.
3. $x = 2 + 2\sec\theta$, $y = 1 + 4\tan\theta$, $\theta \in (-\pi/2, \pi/2) \bigcup (\pi/2, 3\pi/2)$.
4. $x = t$, $y = 4 - t$, $0 \le t \le 4$.

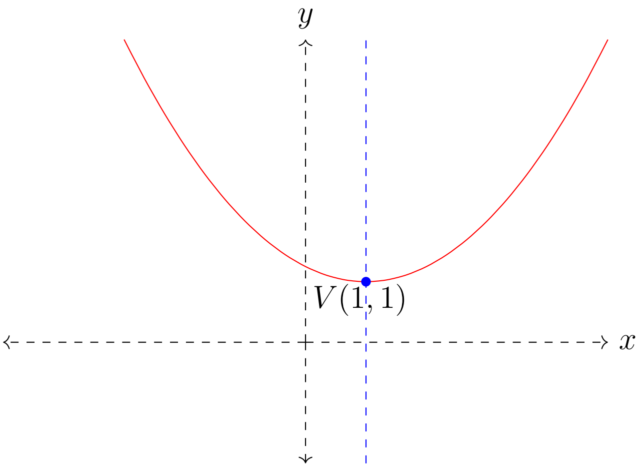
Solutions. (1)

$$x = 2t + 1 \Rightarrow t = \frac{x-1}{2} \text{ and } y = t^2 + 1 \Rightarrow t^2 = y - 1$$

Therefore,

$$y - 1 = t^2 = \left(\frac{x-1}{2}\right)^2 \quad \Rightarrow (y-1) = \frac{1}{4}(x-1)^2 \text{ or } (x-1)^2 = 4(y-1)$$

Shifting the origin at $(1,1)$ we get $X^2 = 4Y$, which the equation of a parabola.



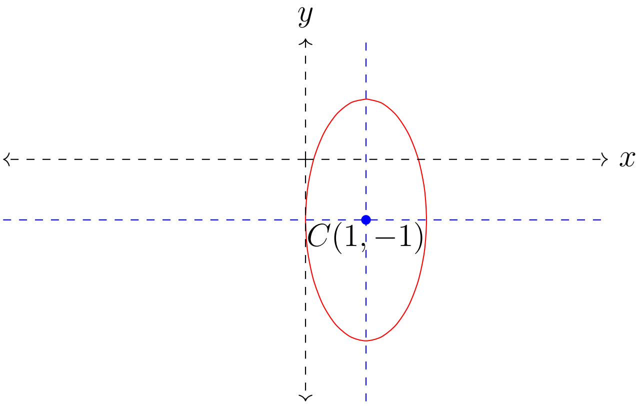
(2)

$$x = 1 + \sin\theta \Rightarrow \sin\theta = (x-1) \text{ and } y = -1 + 2\cos\theta \Rightarrow \cos\theta = \frac{y+1}{2}$$

Since $\sin^2\theta + \cos^2\theta = 1$ we have

$$(x-1)^2 + \frac{(y+1)^2}{4} = 1$$

which is the equation of an ellipse in standard form 2 (major axis parallel to y-axis) with center at $(1, -1)$, $a = 1$ and $b = 2$.



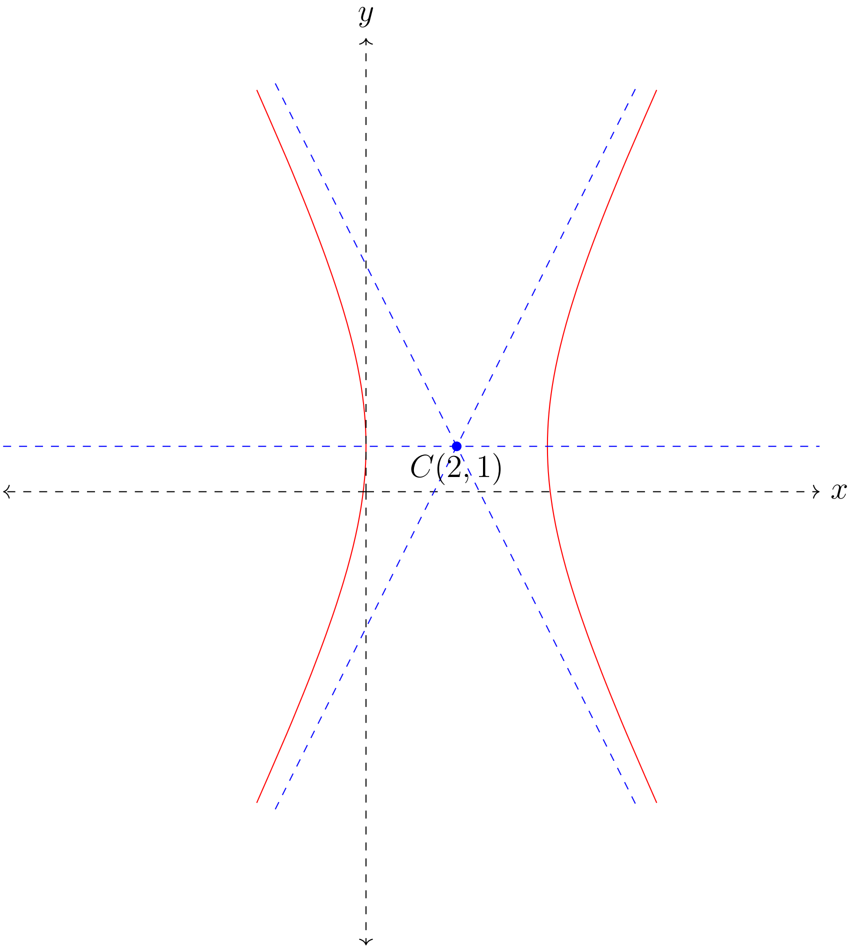
(3)

$$x = 2 + 2\sec\theta \Rightarrow \sec\theta = \frac{x-2}{2} \text{ and } y = 1 + 4\tan\theta \Rightarrow \tan\theta = \frac{y-1}{4}$$

Since $\sec^2\theta - \tan^2\theta = 1$ we have

$$\frac{(x-2)^2}{4} - \frac{(y-1)^2}{16} = 1$$

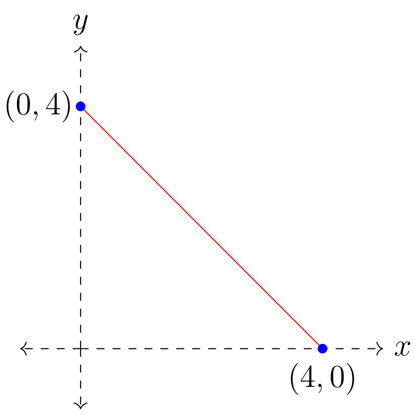
which is the equation of a hyperbola in standard form 1 (axis parallel to x-axis) with center at $(2, 1)$, $a = 2$ and $b = 4$.



(4)

$$x = t \text{ and } y = 4 - t \Rightarrow y = 4 - x \text{ or } x + y = 4$$

which is cartesian equation of a straight line. But since the parameter t varies between $0 \le t \le 4$, the given parametric equation represents a line segment as shown in the graph:-



□

Problem 2: Eliminate the parameter to find the cartesian equation for the following parametric curves.

1. $x = \sqrt{t}$, $y = 1 - t$.
2. $x = t^2$, $y = \ln t$.
3. $x = t^2$, $y = t^3$.

Solutions. (1)

$$x = \sqrt{t} \Rightarrow t = x^2$$

Substituting the value of t in $y = 1 - t$ we get $y = 1 - x^2$ or $\boxed{x^2 + y = 1}$ which is the required cartesian equation.

(2)

$$y = \ln t \Rightarrow t = e^y$$

Substituting the value of t in $x = t^2$ we get:-

$$\boxed{x = e^{2y}.}$$

(3)

$$x = t^2 \Rightarrow x^3 = t^6 \text{ and } y = t^3 \Rightarrow y^2 = t^6$$

Therefore,

$$\boxed{x^3 = y^2}$$

is the required cartesian equation. □

Problem 3: Find the parametric equation of the following conic sections.

1. A parabola with vertex at $(2, 2)$ and focus at $(3, 2)$.
2. An ellipse with center at $(-1, 4)$, a vertex at $(-1, 0)$ and a focus at $(-1, 6)$
3. A hyperbola with foci at $(2, 0)$, $(2, 8)$ and asymptotes $y = 3 + \frac{1}{2}x$, $y = 5 - \frac{1}{2}x$.

Solutions. (1) Shift the origin to the vertex $(2, 2)$ to obtain new coordinates

$$X = x - 2 \text{ and } Y = y - 2$$

In the new coordinates, the focus is at $(3 - 2, 2 - 2) = (1, 0)$. Thus, $p = 1$.
Since the focus lie on +ve X-axis, the parametric equation is given by

$$X = 1t^2,\ Y = 2(1)t \text{ or } x - 2 = t^2,\ y - 2 = 2t$$

that is

$$\boxed{x = 2 + t^2,\ y = 2 + 2t \text{ where } t \in \mathbb{R}.}$$

(2) Shift the origin at the center $(-1, 4)$ to obtain new coordinates

$$X = x + 1 \text{ and } Y = y - 4$$

In the new coordinates, a vertex is at $(-1+1, 0-4) = (0, -4)$
and a focus is at $(-1+1, 6-4) = (0, 2)$.

Therefore, $a = 4$ and $c = 2$. This implies,

$$b^2 = a^2 - c^2 = 4^2 - 2^2 = 16 - 4 = 12 \Rightarrow b = \sqrt{12} = 2\sqrt{3}$$

Since focus and vertex lie on Y-axis, the parametric equation of this ellipse is given by:-

$$X = b\cos\theta,\ Y = a\sin\theta \Rightarrow x + 1 = 2\sqrt{3}\cos\theta,\ y - 4 = 4\sin\theta$$

that is

$$\boxed{x = -1 + 2\sqrt{3}\cos\theta,\ y = 4 + 4\sin\theta \text{ where } \theta \in [0, 2\pi).}$$

(3) We first compute intersection of the asymptotes $y = 3 + \frac{1}{2}x$, $y = 5 - \frac{1}{2}x$ to find center.

$$3 + \frac{1}{2}x = 5 - \frac{1}{2}x \Rightarrow x = 5 - 3 = 2 \Rightarrow y = 3 + \frac{1}{2} \times 2 = 4$$

Therefore, center of the hyperbola lies at $(2, 4)$. Shift the origin at $(2, 4)$ to obtain new coordinates

$$X = x - 2 \text{ and } Y = y - 4$$

The foci in new coordinates are at

$$(2-2, 0-4) = (0, -4) \text{ and } (2-2, 8-4) = (0, 4)$$

Therefore,

$$c = 4 \Rightarrow c^2 = a^2 + b^2 = 16 \ldots\ldots\ldots\ldots (*)$$

Since the foci lie on Y-axis, the slope of asymptotes is given by $\pm\dfrac{a}{b} = \pm\dfrac{1}{2}$. Thus, $b = 2a$.
Substituting $b = 2a$ in $(*)$ we have

$$a^2 + (2a)^2 = 16 \Rightarrow a^2 = \frac{16}{5} \Rightarrow a = \frac{4}{\sqrt{5}} \text{ and } b = \frac{8}{\sqrt{5}}$$

Since the foci are on Y-axis, the parametric equation of the hyperbola is given by:-

$$Y = a\sec\theta,\ X = b\tan\theta \Rightarrow y - 4 = \frac{4}{\sqrt{5}}\sec\theta,\ x - 2 = \frac{8}{\sqrt{5}}\tan\theta$$

Thus, the parametric equation of the given hyperbola is:-

$$\boxed{x = 2 + \frac{8}{\sqrt{5}}\tan\theta,\ y = 4 + \frac{4}{\sqrt{5}}\sec\theta \text{ where } \theta \in [0, 2\pi) \setminus \{\pi/2, 3\pi/2\}.}$$

$\square$