Problem 1: Find the Cartesian coordinates of points whose polar coordinates are as follows:-

$$(3, -\pi/3) \quad , \quad (-2, 3\pi/2) \quad , \quad (-1, 5\pi/4)$$

Solution.

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

Therefore,

$$(3, -\pi/3) \equiv \big(3\cos(-\pi/3), 3\sin(-\pi/3)\big) = \left(\frac{3}{2}, -\frac{3\sqrt{3}}{2}\right)$$
$$(-2, 3\pi/2) \equiv \big(-2\cos(3\pi/2), -2\sin(3\pi/2)\big) = (0, 2)$$
$$(-1, 5\pi/4) \equiv \big(-1\cos(5\pi/4), -1\sin(5\pi/4)\big) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

□

Problem 2: Find the polar coordinates of points whose Cartesian coordinates are as follows:-

$$(-4, 4) \quad , \quad (\sqrt{3}, -1) \quad , \quad (-6, 0)$$

Solution.

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

For $(-4, 4)$,

$$r = \sqrt{(-4)^2 + (4)^2} = 4\sqrt{2} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{4}{-4}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

Therefore, $(-4, 4) \equiv (4\sqrt{2}, 3\pi/4)$.

For $(\sqrt{3}, -1)$,

$$r = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2 \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$$

Therefore, $(\sqrt{3}, -1) \equiv (2, 11\pi/6)$.

For $(-6, 0)$,

$$r = \sqrt{(-6)^2 + (0)^2} = 6 \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{0}{-6}\right) = \pi - 0 = \pi$$

Therefore, $(-6, 0) \equiv (6, \pi)$. □

Problem 3: Identify the curves by finding their Cartesian equations.

1. $r = 4\sec\theta$
2. $r = 5\cos\theta$
3. $r^2\cos 2\theta = 1$

Solution.

$$r = \sqrt{x^2+y^2} \quad , \quad \cos\theta = \frac{x}{\sqrt{x^2+y^2}} \quad , \quad \sin\theta = \frac{y}{\sqrt{x^2+y^2}}$$

(1)

$$r = 4\sec\theta \Rightarrow r\cos\theta = 4 \Rightarrow \sqrt{x^2+y^2}\frac{x}{\sqrt{x^2+y^2}} = 4 \Rightarrow x = 4$$

Therefore, the given equation represents a vertical straight line passing through $(4,0)$.

(2)

$$r = 5\cos\theta \Rightarrow \sqrt{x^2+y^2} = 5\frac{x}{\sqrt{x^2+y^2}} \Rightarrow (x^2+y^2) = 5x$$

$$\Rightarrow x^2 - 5x + y^2 = 0 \Rightarrow \underbrace{x^2 - 2\left(\frac{5}{2}\right)x + \left(\frac{5}{2}\right)^2} - \left(\frac{5}{2}\right)^2 + y^2 = 0$$

$$\Rightarrow \left(x - \frac{5}{2}\right)^2 + y^2 = \left(\frac{5}{2}\right)^2$$

Therefore, the given equation represents a circle with centre at $(2.5, 0)$ and radius 2.5.

(3)

$$r^2\cos 2\theta = 1 \Rightarrow r^2(\cos^2\theta - \sin^2\theta) = 1 \Rightarrow (r\cos\theta)^2 - (r\sin\theta)^2 = 1$$
$$\Rightarrow x^2 - y^2 = 1$$

Therefore, the given equation represents a hyperbola with center at $(0,0)$, axis parallel to x-axis and with $a = b = 1$. □

Problem 4: Find a polar equation of the curve whose Cartesian equation is as follows:-

1. $4y^2 = x$ (a parabola)
2. $x^2 + 4y^2 - 2x = 3$ (an ellipse)
3. $x^2 + y^2 = 2x$ (a circle)

Solution.

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

(1)

$$4y^2 = x \Rightarrow 4(r\sin\theta)^2 = r\cos\theta \Rightarrow 4r^2\sin^2\theta = r\cos\theta$$

So either $r = 0$ or $4r\sin^2\theta = \cos\theta$. But for $\theta = \pi/2$, the point $(0, \pi/2)$ (representing the pole) satisfies $4r\sin^2\theta = \cos\theta$. Therefore, the polar equation of the given parabola is

$$\boxed{4r\sin^2\theta = \cos\theta}$$

(2)

$$x^2 + 4y^2 - 2x = 3 \Rightarrow (r\cos\theta)^2 + 4(r\sin\theta)^2 - 2r\cos\theta = 3$$
$$\Rightarrow r^2(\cos^2\theta + 4\sin^2\theta) - 2r\cos\theta = 3 \Rightarrow r^2(1 + 3\sin^2\theta) - 2r\cos\theta = 3$$

Therefore, the polar equation of the given ellipse is

$$\boxed{r^2(1 + 3\sin^2\theta) - 2r\cos\theta = 3}$$

(3)

$$x^2 + y^2 = 2x \Rightarrow (r\cos\theta)^2 + (r\sin\theta)^2 = 2r\cos\theta$$
$$\Rightarrow r^2(\sin^2\theta + \cos^2\theta) = 2r\cos\theta \Rightarrow r^2 = 2r\cos\theta$$

So either $r = 0$ or $r = 2\cos\theta$. But the point $(0, \pi/2)$ (representing the pole) satisfies the equation $r = 2\cos\theta$. Therefore, the polar equation of the given circle is

$$\boxed{r = 2\cos\theta}$$

□

Problem 5: Evaluate the following expressions and write your answers in the form $a + bi$.

1. $\dfrac{1+i}{1-i}$
2. $\overline{2i(1-i)}$
3. i^{103}
4. $\sqrt{-3}\sqrt{-12}$

Solution. (1)

$$\frac{1+i}{1-i} = \frac{1+i}{1-i} \times \frac{1+i}{1+i} = \frac{(1+i)(1+i)}{(1-i)(1+i)} = \frac{1+i+i+i^2}{1-i^2} = \frac{1+2i-1}{1-(-1)} = \frac{2i}{2} = i$$

(2)

$$\overline{2i(1-i)} = \overline{2i - 2i^2} = \overline{2i - 2(-1)} = \overline{2i + 2} = \overline{2 + 2i} = 2 - 2i$$

(3)

$$i^4 = 1 \Rightarrow i^{103} = i^{100+3} = i^{4\times 25+3} = \left(i^4\right)^{25} i^3 = (1)^{25} i^3 = i^3 = i^2 \times i = (-1)i = -i$$

(4)

$$\sqrt{-3}\sqrt{-12} = \sqrt{3}i \times \sqrt{12}i = \sqrt{36}i^2 = 6i^2 = 6(-1) = -6$$

□