Problem 1: Let $z = 2\sqrt{3} - 2i$ and $w = -1 + i$. Find polar forms of zw, z/w and $1/z$ by putting z and w into polar forms.

Solution. The polar forms of z, w are given by

$$z = |z|(\cos\theta_1 + i\sin\theta_1) \quad , \quad w = |w|(\cos\theta_2 + i\sin\theta_2)$$

where $\theta_1 = \arg z$ and $\theta_2 = \arg w$.

For $z = 2\sqrt{3} - 2i$

$$|z| = \sqrt{(2\sqrt{3})^2 + (-2)^2} = 4 \quad \text{and} \quad \theta_1 = \tan^{-1}\left(\frac{-2}{2\sqrt{3}}\right) = -\frac{\pi}{6}$$

For $w = w = -1 + i$,

$$|w| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2} \quad \text{and} \quad \theta_2 = \tan^{-1}\left(\frac{1}{-1}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

Then

$$zw = |z||w|\big(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\big) \quad \text{and} \quad \frac{z}{w} = \frac{|z|}{|w|}\big(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\big)$$

$$\theta_1 + \theta_2 = -\frac{\pi}{6} + \frac{3\pi}{4} = \frac{7\pi}{12} \quad \text{and} \quad \theta_1 - \theta_2 = -\frac{\pi}{6} - \frac{3\pi}{4} = -\frac{11\pi}{12} = 2\pi - \frac{11\pi}{12} = \frac{13\pi}{12}$$

Therefore,

$$zw = 4\sqrt{2}\big(\cos(7\pi/12) + i\sin(7\pi/12)\big) \quad \text{and} \quad \frac{z}{w} = 2\sqrt{2}\big(\cos(13\pi/12) + i\sin(13\pi/12)\big)$$

$$\frac{1}{z} = \frac{1}{|z|}\big(\cos(-\theta_1) + i\sin(-\theta_1)\big) = \frac{1}{4}\big(\cos(\pi/6) + i\sin(\pi/6)\big)$$

□

Problem 2: Use De Moivre's Theorem to find a and b where $a + bi = (1 - \sqrt{3}i)^5$.

Solutions. Find the polar form of $z = 1 - \sqrt{3}i$ first.

$$|z| = \sqrt{(1)^2 + (\sqrt{3})^2} = 2 \quad \text{and} \quad \arg(z) = \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right) = -\frac{\pi}{3}$$

By De Moivre's theorem,

$$z^n = |z|^n\big(\cos(n\theta) + i\sin(n\theta)\big)$$

Therefore,

$$(1 - \sqrt{3}i)^5 = 2^5\big(\cos(5 \times \frac{-\pi}{3}) + i\sin(5 \times \frac{-\pi}{3})\big) = 32\big(\cos(-\frac{5\pi}{3}) + i\sin(-\frac{5\pi}{3})\big)$$

Now,

$$\cos\left(-\frac{5\pi}{3}\right) = \cos\left(\frac{5\pi}{3}\right) = \cos\left(\frac{6\pi - \pi}{3}\right) = \cos\left(2\pi - \frac{\pi}{3}\right) = \cos\left(-\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$\sin\left(-\frac{5\pi}{3}\right) = -\sin\left(\frac{5\pi}{3}\right) = -\sin\left(\frac{6\pi-\pi}{3}\right) = -\sin\left(2\pi - \frac{\pi}{3}\right) = -\sin\left(-\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

So, we have

$$a+bi = 32\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 16 + 16\sqrt{3}i$$

Hence, $a = 16$ and $b = 16\sqrt{3}$.

$\square$

Problem 3: Find all solutions of the equation $x^2+2x+5=0$.

Solution. By the quadratic formula,

$$x = \frac{-2 \pm \sqrt{2^2 - 4.1.5}}{2.1} = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

So, the given equation has two solutions, namely, $-1+2i$ and $-1-2i$.

Alternatively, one can use completion of squares,

$$x^2+2x+5=0 \Rightarrow \underbrace{x^2+2x+1}_{(x+1)^2} + 4 = 0 \Rightarrow (x+1)^2 = -4 \Rightarrow x+1 = \pm 2i \Rightarrow x = -1 \pm 2i$$

$\square$

Problem 4: Find all the cube roots of i and sketch them in the complex plane.

Solutions.

$$i = 1\big(\cos(\pi/2) + i\sin(\pi/2)\big)$$

So, we need to solve the equation

$$z^3 = 1\big(\cos(\pi/2) + i\sin(\pi/2)\big)$$

Let $z = r\big(\cos\theta + i\sin\theta\big)$. Then

$$r^3\big(\cos(3\theta) + i\sin(3\theta)\big) = 1\big(\cos(\pi/2) + i\sin(\pi/2)\big)$$

$$\Rightarrow r^3 = 1 \quad \text{and} \quad 3\theta = 2k\pi + \frac{\pi}{2} = (4k+1)\frac{\pi}{2}$$

$$\Rightarrow r = 1 \quad \text{and} \quad \theta = (4k+1)\frac{\pi}{6} \quad \text{for } k = 0, \pm1, \pm2, \pm3, \cdots$$

But distinct values occur only for $k = 0, 1, 2$. So, we have

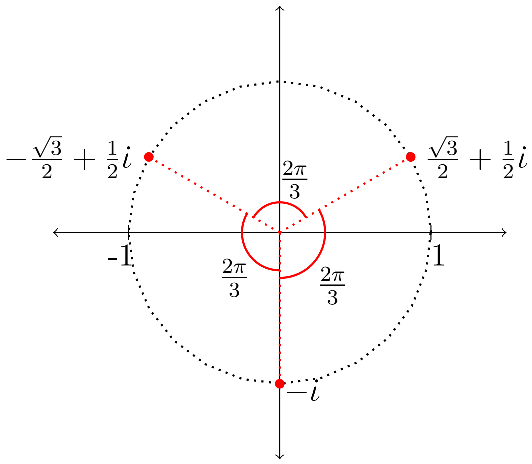
$$z = \cos(\pi/6) + i\sin(\pi/6) \quad \text{or} \quad z = \cos(5\pi/6) + i\sin(5\pi/6) \quad \text{or} \quad z = \cos(9\pi/6) + i\sin(9\pi/6)$$

$$\cos(5\pi/6) = -\cos(\pi/6) = -\sqrt{3}/2 \quad \text{and} \quad \sin(5\pi/6) = \sin(\pi/6) = 1/2$$

$$\cos(9\pi/6) = \cos(3\pi/2) = 0 \quad \text{and} \quad \sin(9\pi/6) = \sin(3\pi/2) = -1$$

Therefore, the three cube roots of i are

$$\frac{\sqrt{3}}{2} + \frac{1}{2}i \quad , \quad -\frac{\sqrt{3}}{2} + \frac{1}{2}i \quad , \quad -i$$



□

Problem 5: Write the following numbers in the form $a + bi$.

$$e^{i\pi/3} \quad , \quad e^{-i\pi} \quad , \quad e^{2+i\pi}$$

Solutions. By Euler's Formula,

$$e^{i\theta} = \cos\theta + i\sin\theta$$

So, we have

$$e^{i\pi/3} = \cos(\pi/3) + i\sin(\pi/3) = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$e^{i\pi} = \cos(-\pi) + i\sin(-\pi) = -1 + 0i = -1$$

$$e^{2+i\pi} = e^2.e^{i\pi} = e^2\big(\cos(\pi) + i\sin(\pi)\big) = e^2(-1+0i) = -e^2$$

□