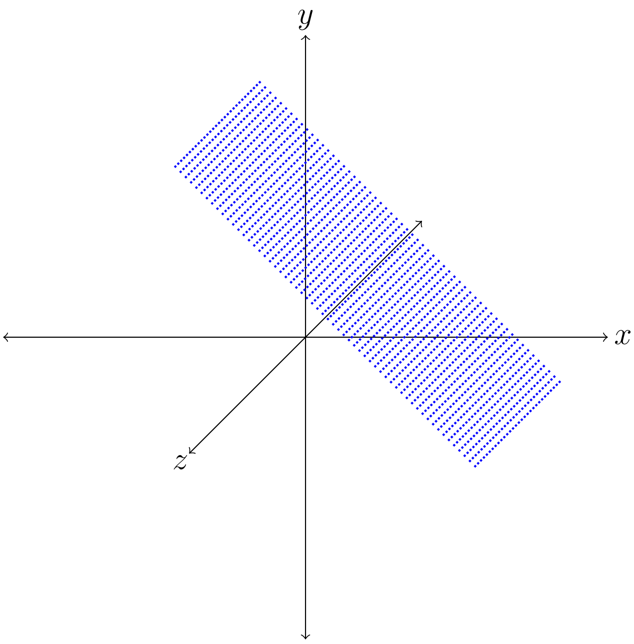
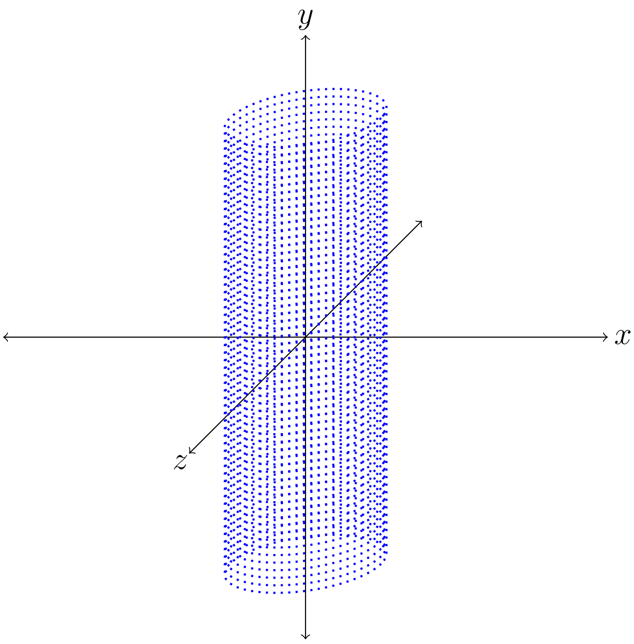
Problem 1: Describe and sketch the surface in $\mathbb{R}^3$ represented by the following equations:-

1. $x + y = 2$
2. $x^2 + z^2 = 9$
3. $x^2 + y^2 + z^2 - 2x - 2z - 2 = 0$

Solution. (1) $x + y = 2$ is the equation of a plane that intersects the x-axis at $(2, 0, 0)$, the y-axis at $(0, 2, 0)$ and does not intersect the z-axis at all.



(2) $x^2 + z^2 = 9$ is the equation of a cylinder of radius 3 and axis being the y-axis.

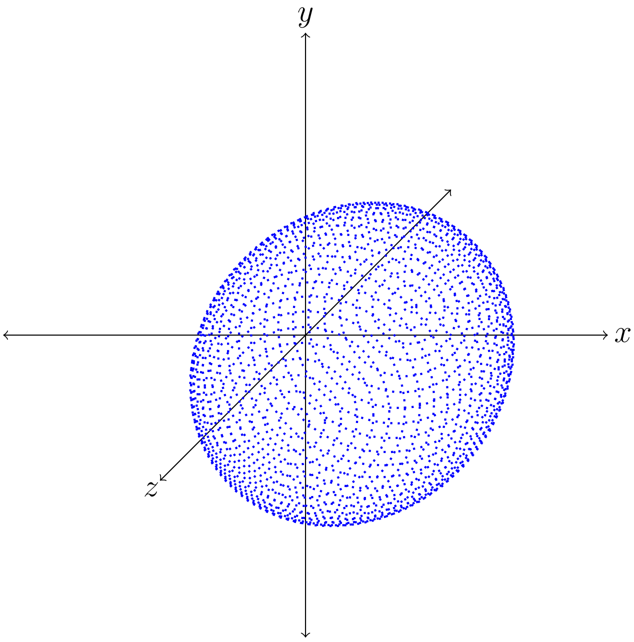


(3) Use completion of squares to bring in standard form.

$$x^2 + y^2 + z^2 - 2x - 2z - 2 = 0 \Rightarrow (\underbrace{x^2 - 2x + 1}_{(x-1)^2} - 1) + y^2 + (\underbrace{z^2 - 2z + 1}_{(z-1)^2} - 1) - 2 = 0$$

$$\Rightarrow (x-1)^2 + y^2 + (z-1)^2 = 4$$

This is the equation of a sphere with radius 2 and center at $(1, 0, 1)$.



$\square$

Problem 2: Find the equation of a sphere centered at $(0, 0, 1)$ and passing through the origin.

Solution. Since the sphere passes through origin, the distance between origin and the center $(0, 0, 1)$ is the radius. Therefore,

$$r = \sqrt{(0-0)^2 + (0-0)^2 + (1-0)^2} = 1$$

Then the equation of the given sphere is

$$(x-0)^2 + (y-0)^2 + (z-1)^2 = 1^2 \Rightarrow x^2 + y^2 + z^2 - 2z + 1 = 1$$

which is

$$\boxed{x^2 + y^2 + z^2 - 2z = 0}$$

$\square$

Problem 3: Let $\vec{a} = 4\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b}$ be the vector from $A(0, 3, 1)$ to $B(2, 3, -1)$.

1. Find the components of $\vec{b}$ and write it in the form $x\hat{i} + y\hat{j} + z\hat{k}$.
2. Find $4\vec{a} - 3\vec{b}$ and $|\vec{a} - \vec{b}|$.
3. Find the vector that has the same direction as $\vec{b}$ but has length 4.
4. Find the unit vector in the direction of $\vec{b} - \vec{a}$.

Solution. (1) $\vec{b} = (2-0)\hat{i} + (3-3)\hat{j} + (-1-1)\hat{k} = 2\hat{i} - 2\hat{k}$

Therefore, the components are $b_x = 2$, $b_y = 0$, $b_z = -2$.

(2)

$$4\vec{a} - 3\vec{b} = 4(4\hat{i} + 3\hat{j} - \hat{k}) - 3(2\hat{i} - 2\hat{k}) = 16\hat{i} + 12\hat{j} - 4\hat{k} - 6\hat{i} + 6\hat{k} = 10\hat{i} + 12\hat{j} + 2\hat{k}$$

$$\vec{a} - \vec{b} = 4\hat{i} + 3\hat{j} - \hat{k} - (2\hat{i} - 2\hat{k}) = 2\hat{i} + 3\hat{j} + \hat{k} \Rightarrow |\vec{a} - \vec{b}| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$$

(3) The vector with length 4 and direction same as $\vec{b}$ is 4 times the unit vector in the direction of $\vec{b}$. Thus, such a vector is given by

$$4\frac{\vec{b}}{|\vec{b}|} = 4\frac{2\hat{i} - 2\hat{k}}{\sqrt{(2)^2 + (-2)^2}} = 4\frac{2\hat{i} - 2\hat{k}}{\sqrt{8}} = \frac{4}{\sqrt{8}}(2\hat{i} - 2\hat{k}) = \frac{4}{2\sqrt{2}}(2\hat{i} - 2\hat{k}) = \frac{4}{\sqrt{2}}\hat{i} - \frac{4}{\sqrt{2}}\hat{k} = 2\sqrt{2}\,\hat{\imath} - 2\sqrt{2}\,\hat{k}$$

(4)

$$\vec{b} - \vec{a} = 2\hat{i} - 2\hat{k} - (4\hat{i} + 3\hat{j} - \hat{k}) = -2\hat{i} - 3\hat{j} - \hat{k} \Rightarrow |\vec{b} - \vec{a}| = \sqrt{(-2)^2 + (-3)^2 + (-1)^2} = \sqrt{14}$$

The unit vector in the direction of $\vec{b} - \vec{a}$ is then given by

$$\frac{\vec{b} - \vec{a}}{|\vec{b} - \vec{a}|} = \frac{-2\hat{i} - 3\hat{j} - \hat{k}}{\sqrt{14}} = -\frac{2}{\sqrt{14}}\,\hat{i} - \frac{3}{\sqrt{14}}\,\hat{j} - \frac{1}{\sqrt{14}}\,\hat{k}$$

□

Problem 4: Let $\vec{a} = \hat{i} + \hat{j}$ and $\vec{b} = \hat{k} - \hat{j}$.

1. Compute $\vec{a}.\vec{b}$ and find the angle between $\vec{a}$ and $\vec{b}$.
2. Find the direction cosines and direction angles of the vector $\vec{a} - \vec{b}$.
3. Find the scalar and vector projections of $\vec{a} + \vec{b}$ onto $\vec{b}$.
4. Find the unit vector orthogonal to $\vec{a}$ and parallel to $\vec{b}$.

Solutions. (1) $\vec{a}.\vec{b} = a_x b_x + a_y b_y + a_z b_z = (1)(0) + (1)(-1) + (0)(1) = -1$

$|\vec{a}| = \sqrt{(1)^2 + (1)^2 + (0)^2} = \sqrt{2}$ and $|\vec{b}| = \sqrt{(0)^2 + (-1)^2 + (1)^2} = \sqrt{2}$

The angle θ between $\vec{a}$ and $\vec{b}$ is given by

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \frac{-1}{\sqrt{2}\sqrt{2}} = -\frac{1}{2} \Rightarrow \theta = \pi - \cos^{-1}\left(\frac{1}{2}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

(2) The direction cosines of a vector $\vec{p}$ are given by

$$\cos\alpha = \frac{p_x}{|\vec{p}|} \quad , \quad \cos\beta = \frac{p_y}{|\vec{p}|} \quad , \quad \cos\gamma = \frac{p_z}{|\vec{p}|}$$

Now, $\vec{a} - \vec{b} = (\hat{i} + \hat{j}) - (\hat{k} - \hat{j}) = \hat{i} + 2\hat{j} - \hat{k}$ and $|\vec{a} - \vec{b}| = \sqrt{(1)^2 + (2)^2 + (-1)^2} = \sqrt{6}$.

Therefore, the direction cosines of $\vec{a} - \vec{b}$ are given by

$$\cos\alpha = \frac{1}{\sqrt{6}} \quad , \quad \cos\beta = \frac{2}{\sqrt{6}} \quad , \quad \cos\gamma = \frac{-1}{\sqrt{6}}$$

Then the direction angles would be

$$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{6}}\right) \quad , \quad \beta = \cos^{-1}\left(\frac{2}{\sqrt{6}}\right) \quad , \quad \gamma = \pi - \cos^{-1}\left(\frac{1}{\sqrt{6}}\right)$$

(3) The scalar projection of a vector $\vec{p}$ onto a vector $\vec{q}$ is given by

$$\text{comp}_{\vec{q}}\vec{p} = \vec{p}.\hat{q}$$

where $\hat{q}$ is the unit vector in the direction of $\vec{q}$. The vector projection of $\vec{p}$ onto $\vec{q}$ is given by

$$\text{proj}_{\vec{q}}\vec{p} = (\vec{p}.\hat{q})\hat{q}$$

Now $\vec{p} = \vec{a} + \vec{b} = (\hat{\imath} + \hat{\jmath}) + (\hat{k} - \hat{\jmath}) = \hat{\imath} + \hat{k}$ and $\hat{q} = \dfrac{\vec{b}}{|\vec{b}|} = \dfrac{1}{\sqrt{2}}(\hat{k} - \hat{\jmath})$.

The scalar projection of $\vec{a} + \vec{b}$ onto $\vec{b}$ is then given by

$$\vec{p}.\hat{q} = (\hat{\imath} + \hat{k}).\frac{1}{\sqrt{2}}(\hat{k} - \hat{\jmath}) = \frac{1}{\sqrt{2}}\big((1)(0) + (0)(-1) + (1)(1)\big) = \frac{1}{\sqrt{2}}$$

The vector projection of $\vec{a} + \vec{b}$ onto $\vec{b}$ is then given by

$$(\vec{p}.\hat{q})\hat{q} = \frac{1}{\sqrt{2}}\hat{q} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}(\hat{k} - \hat{\jmath}) = \frac{1}{2}(\hat{k} - \hat{\jmath}).$$

(4) Let $\vec{p}$ be the unit vector orthogonal to $\vec{a}$ and parallel to $\vec{b}$.

Since $\vec{p}$ is parallel to $\vec{b}$, it has to be proportional to $\vec{b}$. Therefore,

$$\vec{p} = c\,\hat{k} - c\,\hat{\jmath}$$

for some scalar $c \in \mathbb{R}$.

Since $\vec{p}$ is orthogonal to $\vec{a}$, we must have $\vec{p}.\vec{a} = 0$. Therefore,

$$(c\,\hat{k} - c\,\hat{\jmath}).(\hat{\imath} + \hat{\jmath}) = 0 \Rightarrow (0)(1) + (-c)(1) + (c)(0) = 0 \Rightarrow -c = 0 \Rightarrow c = 0$$

Thus, we must have $\vec{p} = \vec{0}$, but then $|\vec{p}| = \sqrt{0^2 + 0^2 + 0^2} = 0$ and $\vec{p}$ cannot be a unit vector.

Hence, there is no such vector which is orthogonal to $\vec{a}$ and parallel to $\vec{b}$. $\square$

Problem 5: Find the following vectors, without using determinant, but by using the properties of cross products.

1. $(\hat{i} \times \hat{j}) \times \hat{k}$
2. $(\hat{i} + 2\hat{j}) \times (\hat{i} - \hat{j} + 2\hat{k})$

Problem 6: Let $P(0, -2, 0)$, $Q(4, 1, -2)$, $R(5, 3, 1)$ be points in the 3-D space.

1. Find the area of the triangle PQR.
2. Find a nonzero vector orthogonal to the plane passing through points P, Q and R.

Problem 7: Find the volume of the parallelepiped determined by the vectors

$$\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\vec{b} = -\hat{i} + \hat{j} + 2\hat{k}$$

$$\vec{c} = 2\hat{i} + \hat{j} + 4\hat{k}$$