Problem 1: Find the following vectors, without using determinant, but by using the properties of cross products.

1. $(\hat{i} \times \hat{j}) \times \hat{k}$
2. $(\hat{i} + 2\hat{j}) \times (\hat{i} - \hat{j} + 2\hat{k})$

Solutions. (1) We know that $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$. Therefore,

$$(\hat{i} \times \hat{j}) \times \hat{k} = -\hat{k} \times (\hat{i} \times \hat{j}) = -\big((\hat{k}.\hat{j})\hat{i} - (\hat{k}.\hat{i})\hat{j}\big) = -(0\hat{i} - 0\hat{j}) = \vec{0}$$

Alternatively, since $\hat{i} \times \hat{j} = \hat{k}$, we have

$$(\hat{i} \times \hat{j}) \times \hat{k} = \hat{k} \times \hat{k} = \vec{0}.$$

(2) We use distributivity of cross product over addition.

$$\begin{aligned}(\hat{i} + 2\hat{j}) \times (\hat{i} - \hat{j} + 2\hat{k}) &= \hat{i} \times (\hat{i} - \hat{j} + 2\hat{k}) + 2\hat{j} \times (\hat{i} - \hat{j} + 2\hat{k}) \\ &= \hat{i} \times \hat{i} - \hat{i} \times \hat{j} + 2\,\hat{i} \times \hat{k} + 2\,\hat{j} \times \hat{i} - 2\,\hat{j} \times \hat{j} + 4\,\hat{j} \times \hat{k} \\ &= \vec{0} - \hat{k} + 2\,(-\hat{j}) + 2\,(-\hat{k}) - 2(\vec{0}) + 4\,\hat{i} = \boxed{4\,\hat{i} - 2\,\hat{j} - 3\,\hat{k}}\end{aligned}$$

□

Problem 2: Let $P(0, -2, 0)$, $Q(4, 1, -2)$, $R(5, 3, 1)$ be points in the 3-D space.

1. Find the area of the triangle PQR.
2. Find a nonzero vector orthogonal to the plane passing through points P, Q and R.

Solutions. (1) The area of the triangle PQR is given by

$$A = \frac{1}{2}\,|\vec{a} \times \vec{b}|$$

where $\vec{a}$ is vector from Q to P and $\vec{b}$ is the vector from Q to R.
Note that $\vec{a}$ and $\vec{b}$ can be chosen to be any two adjacent sides of the triangle PQR.

Now, $\vec{a} = \big(4 - 0\big)\,\hat{i} + \big(1 - (-2)\big)\,\hat{j} + \big(-2 - 0\big)\,\hat{k} = 4\,\hat{i} + 3\,\hat{j} - 2\,\hat{k}$ and

$\vec{b} = \big(5 - 4\big)\,\hat{i} + \big(3 - 1\big)\,\hat{j} + \big(1 - (-2)\big)\,\hat{k} = \hat{i} + 2\,\hat{j} + 3\,\hat{k}$.

The cross product $\vec{a} \times \vec{b}$ is given by

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 3 & -2 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ 2 & 3 \end{vmatrix}\,\hat{i} - \begin{vmatrix} 4 & -2 \\ 1 & 3 \end{vmatrix}\,\hat{j} + \begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix}\,\hat{k} = 13\,\hat{i} - 14\,\hat{j} + 5\,\hat{k}$$

Thus, the ares of triangle PQR is given by

$$A = \frac{1}{2}\sqrt{(13)^2 + (-14)^2 + (5)^2} = \frac{1}{2}\sqrt{390}$$

(2) The nonzero vector orthogonal to the plane passing through P, Q, R is proportional to $\vec{a} \times \vec{b}$ where $\vec{a}$ is vector from Q to P and $\vec{b}$ is the vector from Q to R. As in the previous part $\vec{a}$ and $\vec{b}$ can be chosen to be any two adjacent sides of the triangle PQR.

Thus, one such vector is $\vec{a} \times \vec{b} = 13\,\hat{\imath} - 14\,\hat{\jmath} + 5\,\hat{k}$. $\square$

Problem 3: Find the volume of the parallelepiped determined by the vectors

$$\vec{a} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$$

$$\vec{b} = -\hat{\imath} + \hat{\jmath} + 2\hat{k}$$

$$\vec{c} = 2\hat{\imath} + \hat{\jmath} + 4\hat{k}$$

Solutions. The volume of the parallelepiped determined by any three given vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ is given by $|(\vec{a} \times \vec{b}).\vec{c}\,|$. So, the required volume is

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{vmatrix} .(2\,\hat{\imath} + \hat{\jmath} + 4\,\hat{k}) = \begin{vmatrix} 2 & 1 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} + 4\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix}$$

$$= 2(1) - 1(5) + 4(3) = 9$$

$\square$