Problem 1: Find the vector, parametric and Cartesian (also called symmetric) equation of the following lines:-

1. The line that passes through the points $(-8, 1, 4)$ and $(3, -2, 4)$.

   Solution: The vector equation is $\vec{r}(t) = \vec{a} + t\vec{v}$, $t \in \mathbb{R}$, where $\vec{a}$ is the position vector of one of the points and $\vec{v}$ is the direction vector of the line.

   $$\vec{v} = \big(3 - (-8)\big)\hat{\imath} + \big(-2 - 1\big)\hat{\jmath} + \big(4 - 4\big)\hat{k} = 11\,\hat{\imath} - 3\,\hat{\jmath}$$

   Choose the point $(-8, 1, 4)$ for $\vec{a}$. Then

   $$\vec{a} = -8\,\hat{\imath} + \hat{\jmath} + 4\,\hat{k}$$

   so that

   $$\vec{r}(t) = -8\,\hat{\imath} + \hat{\jmath} + 4\,\hat{k} + t(11\,\hat{\imath} - 3\,\hat{\jmath}), \quad t \in \mathbb{R}$$

   Thus, the vector equation of the given line is

   $$\boxed{\vec{r}(t) = (-8 + 11t)\hat{\imath} + (1 - 3t)\hat{\jmath} + 4\hat{k}, \quad t \in \mathbb{R}}$$

   The parametric equation will then be

   $$\boxed{x(t) = -8 + 11t, \quad y(t) = 1 - 3t, \quad z(t) = 4, \quad t \in \mathbb{R}}$$

   And the cartesian equation would be

   $$\boxed{\frac{x + 8}{11} = \frac{y - 1}{-3}, \qquad z = 4}$$

2. The line that passes through the point $(2, 1, 0)$ and is perpendicular to both $\hat{\imath} + \hat{\jmath}$ and $\hat{\jmath} + \hat{k}$.

   Solution: We have one point, that is, $\vec{a} = 2\,\hat{\imath} + \hat{\jmath}$ and direction will be given by the vector that is perpendicular to both $\hat{\imath} + \hat{\jmath}$ and $\hat{\jmath} + \hat{k}$. So,

   $$\vec{v} = \big(\hat{\imath} + \hat{\jmath}\big) \times \big(\hat{\jmath} + \hat{k}\big) = \big(\hat{\imath} \times \hat{\jmath}\big) + \big(\hat{\imath} \times \hat{k}\big) + \big(\hat{\jmath} \times \hat{\jmath}\big) + \big(\hat{\jmath} \times \hat{k}\big) = \hat{k} - \hat{\jmath} + \vec{0} + \hat{\imath}$$

   Thus, $\vec{v} = \hat{\imath} - \hat{\jmath} + \hat{k}$ and we have

   $$\vec{r}(t) = \vec{a} + t\vec{v} = 2\,\hat{\imath} + \hat{\jmath} + t(\hat{\imath} - \hat{\jmath} + \hat{k}) = (2 + t)\,\hat{\imath} + (1 - t)\,\hat{\jmath} + t\,\hat{k}, \quad t \in \mathbb{R}$$

   Thus, the vector equation of the given line is

   $$\boxed{\vec{r}(t) = (2 + t)\,\hat{\imath} + (1 - t)\,\hat{\jmath} + t\,\hat{k}, \quad t \in \mathbb{R}}$$

   The parametric equation will then be

   $$\boxed{x(t) = 2 + t, \quad y(t) = 1 - t, \quad z(t) = t, \quad t \in \mathbb{R}}$$

   And the cartesian equation would be

   $$\boxed{x - 2 = 1 - y = z}$$

3. The line that passes through the point $(-6, 2, 3)$ and is parallel to the line $\dfrac{x}{2} = \dfrac{y}{3} = \dfrac{z+1}{1}$.

   Solution: The required line passes through $\vec{a} = -6\,\hat{\imath} + 2\,\hat{\jmath} + 3\,\hat{k}$ and since it is parallel to the line $\dfrac{x}{2} = \dfrac{y}{3} = \dfrac{z+1}{1}$, its direction is same as that of $\dfrac{x}{2} = \dfrac{y}{3} = \dfrac{z+1}{1}$.
   Hence, we have $\vec{v} = 2\,\hat{\imath} + 3\,\hat{\jmath} + \hat{k}$. So that

   $$\vec{r}(t) = \vec{a} + t\vec{v} = -6\,\hat{\imath} + 2\,\hat{\jmath} + 3\,\hat{k} + t(2\,\hat{\imath} + 3\,\hat{\jmath} + \hat{k}) = (-6+2t)\,\hat{\imath} + (2+3t)\,\hat{\jmath} + (3+t)\,\hat{k}$$

   Thus, the vector equation of the given line is

   $$\boxed{\vec{r}(t) = (-6+2t)\,\hat{\imath} + (2+3t)\,\hat{\jmath} + (3+t)\,\hat{k}, \quad t \in \mathbb{R}}$$

   The parametric equation will then be

   $$\boxed{x(t) = -6+2t, \quad y(t) = 2+3t, \quad z(t) = 3+t, \quad t \in \mathbb{R}}$$

   And the Cartesian equation would be

   $$\boxed{\frac{x+6}{2} = \frac{y-2}{3} = z-3}$$

4. The line that passes through the point $(1, 0, 6)$ and is perpendicular to the plane $x + 3y + z = 5$.

   Solution: The required line passes through $\vec{a} = \hat{\imath} + 6\,\hat{k}$ and since it is perpendicular to the plane $x + 3y + z = 5$, its direction is same as the normal vector of $x + 3y + z = 5$. Hence, $\vec{v} = \hat{\imath} + 3\,\hat{\jmath} + \hat{k}$ and we have

   $$\vec{r}(t) = \hat{\imath} + 6\,\hat{k} + t(\hat{\imath} + 3\,\hat{\jmath} + \hat{k}) = (1+t)\,\hat{\imath} + 3t\,\hat{\jmath} + (6+t)\,\hat{k}$$

   Thus, the vector equation of the given line is

   $$\boxed{\vec{r}(t) = (1+t)\,\hat{\imath} + 3t\,\hat{\jmath} + (6+t)\,\hat{k}, \quad t \in \mathbb{R}}$$

   The parametric equation will then be

   $$\boxed{x(t) = 1+t, \quad y(t) = 3t, \quad z(t) = 6+t, \quad t \in \mathbb{R}}$$

   And the Cartesian equation would be

   $$\boxed{x - 1 = \frac{y}{3} = z - 6}$$

Problem 2: Determine whether the lines L_1 and L_2 are parallel, skew or intersecting. If they intersect, find the point of intersection.

1.
$$L_1 : \frac{x-3}{2} = \frac{y-4}{-1} = \frac{z-1}{3} \quad ; \quad L_2 : \frac{x-1}{4} = \frac{y-3}{-2} = \frac{z-4}{5}$$

Solution: The direction of L_1 is

$$\vec{v}_1 = 2\,\hat{i} - \hat{j} + 3\,\hat{k}$$

and the direction of L_2 is

$$\vec{v}_2 = 4\,\hat{i} - 2\,\hat{j} + 5\,\hat{k}$$

If L_1, L_2 were parallel, we would have some scalar $\alpha \in \mathbb{R}$ such that $\vec{v}_1 = \alpha\,\vec{v}_2$, so that

$$2 = 4\alpha \Rightarrow \alpha = \frac{1}{2}$$

$$-1 = -2\,\alpha \Rightarrow \alpha = \frac{1}{2}$$

$$3 = 5\alpha \Rightarrow \alpha = \frac{3}{5}$$

which is inconsistent. Therefore, L_1 and L_2 are not parallel.

Now in parametric form we have

$L_1 : x(t) = 3{+}2t,\ y(t) = 4{-}t,\ z(t) = 1{+}3t$ and $L_2 : x(s) = 1{+}4s,\ y(s) = 3{-}2s,\ z(s) = 4{+}5s$

Equating x, y and z, we get

$$3 + 2t = 1 + 4s \ldots\ldots (1)$$

$$4 - t = 3 - 2s \ldots\ldots (2)$$

$$1 + 3t = 4 + 5s \ldots\ldots (3)$$

Solving (1) and (2) for t and s we have:- $t = 4 - 3 + 2s = 1 + 2s$ from (2). Substituting this for t in (1) we get $3 + 2(1 + 2s) = 1 + 4s \Rightarrow 3 + 2 + 4s = 1 + 4s \Rightarrow 5 = 1$ which is not possible for any value of t and s. Therefore, the given lines have to be $\boxed{\textit{skew}}$.

2.

$L_1 : x = 5 - 12t, \quad y = 3 + 9t, \quad z = 1 - 3t \quad ; \quad L_2 : x = 3 + 8s, \quad y = -6s, \quad z = 7 + 2s$

Solution: The direction of L_1 is

$$\vec{v}_1 = -12\,\hat{i} + 9\,\hat{j} - 3\,\hat{k}$$

and the direction of L_2 is

$$\vec{v}_2 = 8\,\hat{i} - 6\,\hat{j} + 2\,\hat{k}$$

If L_1, L_2 were parallel, we would have some scalar $\alpha \in \mathbb{R}$ such that $\vec{v}_1 = \alpha\,\vec{v}_2$, so that

$$-12 = 8\alpha \Rightarrow \alpha = -\frac{3}{2}$$

$$9 = -6\,\alpha \Rightarrow \alpha = -\frac{3}{2}$$

$$-3 = 2\alpha \Rightarrow \alpha = -\frac{3}{2}$$

which is consistent. Therefore, the given lines L_1 and L_2 are $\boxed{parallel}$.

3.

$$L_1 : \frac{x-2}{1} = \frac{y-3}{-2} = \frac{z-1}{-3} \quad ; \quad L_2 : \frac{x-3}{1} = \frac{y+4}{3} = \frac{z-2}{-7}$$

Solution: The direction of L_1 is

$$\vec{v}_1 = \hat{\imath} - 2\,\hat{\jmath} - 3\,\hat{k}$$

and the direction of L_2 is

$$\vec{v}_2 = \hat{\imath} + 3\,\hat{\jmath} - 7\,\hat{k}$$

If L_1, L_2 were parallel, we would have some scalar $\alpha \in \mathbb{R}$ such that $\vec{v}_1 = \alpha\,\vec{v}_2$, so that

$$1 = \alpha \Rightarrow \alpha = 1$$

$$-2 = 3\,\alpha \Rightarrow \alpha = -\frac{2}{3}$$

$$-3 = -7\alpha \Rightarrow \alpha = \frac{3}{7}$$

which is inconsistent. Therefore, L_1 and L_2 are not parallel.

Now, in parametric form, we have

$$L_1 : \; x = 2 + t,\; y = 3 - 2t,\; z = 1 - 3t \quad \text{and} \quad L_2 : \; x = 3 + s,\; y = -4 + 3s,\; z = 2 - 7s$$

Equating x, y and z we have

$$2 + t = 3 + s \ldots\ldots (1)$$

$$3 - 2t = -4 + 3s \ldots\ldots (2)$$

$$1 - 3t = 2 - 7s \ldots\ldots (3)$$

Solving (1) and (2) for t and s we have:- $t = 1 + s$ from (1). Substituting this in (2) we get $3-2(1+s) = -4+3s \Rightarrow 1-2s = -4+3s \Rightarrow 5s = 5 \Rightarrow s = 1 \Rightarrow t = 1+s = 1+1 = 2$.

So we get $t = 2$, $s = 1$ from (1) and (2). Now we plug these values of t and s in (3).

$$1 - 3(2) = 2 - 7(1) \Rightarrow 1 - 6 = 2 - 7 \Rightarrow -5 = -5$$

Therefore, the three equations are consistent and the given lines L_1 and L_2 $\boxed{intersect}$.

To find the point of intersection, either put $t = 2$ in parametric equation of L_1 or $s = 1$ in parametric equation of L_2. So we have

$$x = 2 + 2,\ y = 3 - 2(2),\ z = 1 - 3(2) \Rightarrow x = 4,\ y = -1,\ z = -5$$

Hence, the point of intersection of L_1 and L_2 is $\boxed{(4, -1, -5)}$.

Problem 3: Find the vector and Cartesian equation of the following planes.

1. The plane passing through the points $(2, 1, 2)$, $(3, -8, 6)$ and $(-2, -3, 1)$.

   Solution: From these three point we can find two vectors that lie in the plane.

   Choosing one of them to be the vector from $(2, 1, 2)$ to $(3, -8, 6)$ we get

   $$\vec{v}_1 = (3 - 2)\,\hat{i} + (-8 - 1)\,\hat{j} + (6 - 2)\,\hat{k} = \hat{i} - 9\,\hat{j} + 4\,\hat{k}$$

   and the other one to be the vector from $(3, -8, 6)$ to $(-2, -3, 1)$ we get

   $$\vec{v}_2 = (-2 - 3)\,\hat{i} + (-3 + 8)\,\hat{j} + (1 - 6)\,\hat{k} = -5\,\hat{i} + 5\,\hat{j} - 5\hat{k}$$

   Then a normal vector to the plane is given by $\vec{n} = \vec{v}_1 \times \vec{v}_2$. So we have

   $$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -9 & 4 \\ -5 & 5 & -5 \end{vmatrix} = 25\,\hat{i} - 15\,\hat{j} - 40\,\hat{k}$$

   The vector equation of a plane is given by $\vec{r}.\vec{n} = \vec{a}.\vec{n}$ where $\vec{n}$ is a normal vector to the plane and $\vec{a}$ is the position vector of some point in the plane.

   Choosing the first point $(2, 1, 2)$ we have $\vec{a} = 2\,\hat{i} + \hat{j} + 2\,\hat{k}$. Then

   $$\vec{a}.\vec{n} = 50 - 15 - 80 = -45$$

   So we have

   $$\vec{r}.(25\,\hat{i} - 15\,\hat{j} - 40\,\hat{k}) = -45 \Rightarrow \vec{r}.(5\,\hat{i} - 3\,\hat{j} - 8\,\hat{k}) = -9$$

   where in second step, 5 was factored out from both sides and canceled.

   Thus, the vector equation of the given plane is

   $$\boxed{\vec{r}.(5\,\hat{i} - 3\,\hat{j} - 8\,\hat{k}) + 9 = 0}$$

   And the Cartesian equation is

   $$\boxed{5x - 3y - 8z + 9 = 0}$$

2. The plane passing through the point $(2, 0, 1)$ and perpendicular to the line $x = 3t$, $y = 2 - t$, $z = 3 + 4t$.

   Solution: For some point in the plane we have $\vec{a} = 2\,\hat{i} + \hat{k}$. Since the plane is perpendicular to the given line, its normal vector has to be parallel to the direction vector of this line. Now the direction vector of the given line is

   $$\vec{v} = 3\,\hat{i} - \hat{j} + 4\,\hat{k}$$

So, a normal vector to the plane is

$$\vec{n} = 3\,\hat{\imath} - \hat{\jmath} + 4\,\hat{k}$$

We have

$$\vec{a}.\vec{n} = 2(3) + 0(-1) + 1(4) = 10$$

Therefore, the vector equation of the given plane is

$$\boxed{\vec{r}.(3\,\hat{\imath} - \hat{\jmath} + 4\,\hat{k}) = 10}$$

And the Cartesian equation is

$$\boxed{3x - y + 4z = 10}$$

3. The plane passing through the point $(3, -2, 8)$ and parallel to the plane $z = x + y$.

   Solution: We have $\vec{a} = 3\,\hat{\imath} - 2\,\hat{\jmath} + 8\,\hat{k}$. Since the plane is parallel to the plane $z = x + y$, its normal vector is parallel to that of $z = x + y$. Thus, a normal vector to the required plane can be taken to the normal vector of the plane $z = x + y$ or $x + y - z = 0$. Hence,

$$\vec{n} = \hat{\imath} + \hat{\jmath} - \hat{k}$$

   We have

$$\vec{a}.\vec{n} = 3(1) - 2(1) + 8(-1) = -7$$

   Therefore, the vector equation of the plane is

$$\boxed{\vec{r}.(\hat{\imath} + \hat{\jmath} - \hat{k}) = -7}$$

   And the Cartesian equation is

$$\boxed{x + y - z = -7}$$

4. The plane that passes through the point $(3, 5, -1)$ and contains the line $x = 4 - t$, $y = 2t - 1$, $z = -3t$.

   Solution: We have $\vec{a} = 3\,\hat{\imath} + 5\,\hat{\jmath} - \hat{k}$. The plane contains the line

$$L : x = 4 - t \quad y = 2t - 1, \quad z = -3t$$

   which has direction vector $\vec{v} = -\hat{\imath} + 2\,\hat{\jmath} - 3\,\hat{k}$. We also see that L passes through the point $(4, -1, 0)$. Since the line lies in the plane this point and the direction vector $\vec{v}$ both lie in the plane. Thus, the vector joining $(3, 5, -1)$ to $(4, -1, 0)$ also lie in the plane. This vector is given by

$$\vec{w} = (4 - 3)\,\hat{\imath} + (-1 - 5)\,\hat{\jmath} + (0 + 1)\,\hat{k} = \hat{\imath} - 6\,\hat{\jmath} + \hat{k}$$

   Now the normal vector is perpendicular to the plane and hence perpendicular to both $\vec{v}$ and $\vec{w}$ since these vector lie in the plane. Thus,

$$\vec{n} = \vec{v} \times \vec{w} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 2 & -3 \\ 1 & -6 & 1 \end{vmatrix} = -16\,\hat{\imath} - 2\,\hat{\jmath} + 4\,\hat{k}$$

$$\vec{a}.\vec{n} = 3(-16) + 5(-2) + (-1)(4) = -62$$

So, we have

$$\vec{r}.(-16\,\hat{i} - 2\,\hat{j} + 4\,\hat{k}) = -62 \Rightarrow \vec{r}.(8\,\hat{i} + \hat{j} - 2\,\hat{k}) = 31$$

where in the second step -2 was factored out from both the sides and cancelled. Thus the vector equation of the given plane is

$$\boxed{\vec{r}.(8\,\hat{i} + \hat{j} - 2\,\hat{k}) = 31}$$

And the Cartesian equation is given by

$$\boxed{8x + y - 3z = 31}$$

5. The plane passing through the point $(1, 5, 1)$ and perpendicular to the planes $2x + y - 2z = 2$ and $x + 3z = 4$.

   Solution: We have $\vec{a} = \hat{i} + 5\,\hat{j} + \hat{k}$. Since the plane is perpendicular to two given planes, its normal vector is perpendicular to the normal vectors of the two given planes $2x + y - 2z = 2$ and $x + 3z = 4$. The normal vectors of these planes are

$$\vec{n}_1 = 2\,\hat{i} + \hat{j} - 2\,\hat{k}$$

$$\vec{n}_2 = \hat{i} + 3\,\hat{k}$$

   Thus, the normal vector $\vec{n}$ of the required plane is

$$\vec{n} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 1 & 0 & 3 \end{vmatrix} = 3\,\hat{i} - 8\,\hat{j} - \hat{k}$$

   Then $\vec{a}.\vec{n} = 1(3) + 5(-8) + 1(-1) = -38$.

   Therefore, the vector equation of the plane is

$$\boxed{\vec{r}.(3\,\hat{i} - 8\,\hat{j} - \hat{k}) = -38}$$

   And the Cartesian equation is

$$\boxed{3x - 8y - z = -38}$$

Problem 4: Consider the planes $P_1 : 3x - 2y + z = 1$, $P_2 : 2x + y - 3z = 3$ and the line $L : x = 2 - 2t$, $y = -15 - t$, $z = 1 + 4t$.

1. Find the points of intersection of L with P_1 and P_2.

   Solution: <u>Point of intersection of L and P_1</u>:-

   Put $x = 2 - 2t$, $y = -15 - t$, $z = 1 + 4t$ in the equation $3x - 2y + z = 1$ of P_1. We get

$$3(2 - 2t) - 2(-15 - t) + (1 + 4t) = 1 \Rightarrow 6 - 6t + 30 + 2t + 1 + 4t = 1 \Rightarrow 37 = 1$$

   which is inconsistent. Since the coefficient of t becomes 0 on the Left hand side the line L is parallel to P_1. So it can either completely lie in the plane or not intersect P_1 at all. But if it would have been completely lying in the plane we would have got $1 = 1$ which is not the case. Therefore, $\boxed{L \text{ does not intersect } P_1}$.

Point of intersection of L and P_2:-

Put $x = 2 - 2t$, $y = -15 - t$, $z = 1 + 4t$ in the equation $2x + y - 3z = 3$ of P_2. We get

$$2(2-2t)+(-15-t)-3(1+4t) = 3 \Rightarrow 4-4t-15-t-3-12t = 3 \Rightarrow -17t = 17 \Rightarrow t = -1$$

So, the line L intersects P_2 and the point of intersection is given by

$$x = 2 - 2(-1) = 4, \quad y = -15 - (-1) = -14, \quad z = 1 + 4(-1) = -3$$

Thus, the point of intersection of L with P_2 is $\boxed{(4, -14, -3)}$.

2. Find the angle between P_1 and P_2.

   Solution: The normal vectors of P_1 and P_2 are

$$\vec{n}_1 = 3\,\hat{i} - 2\,\hat{j} + \hat{k}$$

$$\vec{n}_2 = 2\,\hat{i} + \hat{j} - 3\,\hat{k}$$

   The angle between P_1, P_2 is same as the angle between $\vec{n}_1$ and $\vec{n}_2$. Thus,

$$\cos\theta = \frac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{3(2) - 2(1) + 1(-3)}{\left(\sqrt{(3)^2 + (-2)^2 + (1)^2}\right)\left(\sqrt{(2)^2 + (1)^2 + (-3)^2}\right)} = \frac{1}{\sqrt{14}\,\sqrt{14}} = \frac{1}{14}$$

   Therefore, the angle between the two given planes is $\boxed{\theta = \cos^{-1}\left(\frac{1}{14}\right)}$.

3. Find the equation of the line of intersection of P_1 and P_2.

   Solution: The line of intersection of P_1 and P_2 is perpendicular to both $\vec{n}_1$ and $\vec{n}_2$. Thus, the direction vector of the line of intersection is

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & 1 \\ 2 & 1 & -3 \end{vmatrix} = 5\,\hat{i} + 11\,\hat{j} + 7\,\hat{k}$$

   To find a point lying on the line of intersection we solve the equations of two given planes for x, y, z. Since there are three variables and just two equations, we let $z = 0$ and then we get

$$3x - 2y = 1 \ldots\ldots (1)$$

$$2x + y = 3 \ldots\ldots (2)$$

   From (2) we get $y = 3-2x$. Putting this in (1) we have $3x-2(3-2x) = 1 \Rightarrow 7x-6 = 1$ $\Rightarrow 7x = 7 \Rightarrow x = 1 \Rightarrow y = 3 - 2x = 3 - 2(1) = 1$. Thus, $(1, 1, 0)$ is point lying on the line of intersection. Now we have the equation of the line of intersection to be

$$\boxed{\frac{x-1}{5} = \frac{y-1}{11} = \frac{z}{7}}$$