Learning Objectives:

1. Understand the intuitive definition of the limit of a function at a given point.
2. The left hand and right hand limits of a function at a given point.
3. Intuitive definition of an infinite limit.
4. What are vertical asymptotes to the graph of a function?

$f(x)$

Consider the expression

$$\lim_{x\to 4} \frac{x^2}{x+4}.$$

$|x-4|$ is decreasing 10^{-k}

x:	4.1	4.01	4.001	4.0001	3.9	3.99	3.999	3.9999
$\frac{x^2}{x+4}$:	2.1	2.01	2.001	2.0001	1.9	1.99	1.999	1.9999

$|f(x)-2|$ is getting smaller.

We see that the values of $f(x) = \dfrac{x^2}{x+4}$ are getting closer and closer to 2 as x approaches 4. We write this as

$$\lim_{x\to 4} \frac{x^2}{x+4} = 2 .$$

Notice that $f(4) = 2$.

$h = |x-a|$

$x \to a$ means that x lies in $(a-h, a) \cup (a, a+h)$

h very small +ve number

Intuitive definition of a limit

Let f be a function defined on both sides of a except possibly at a itself. Suppose that $f(x)$ becomes arbitrarily close to the number L (written as $f(x) \to L$) as x approaches a ($x \to a$). Then we say that the limit of $f(x)$ as x approaches a is L and we write

$$\lim_{x\to a} f(x) = L .$$

Note that in general:

1. The number a may or may not be in the domain of the function f.
2. We may not always have $\lim_{x\to a} f(x) = f(a)$.

Example 1.

Guess the value of $\lim_{x \to 1} \dfrac{x-1}{x^2-1}$.

Note that 1 is not in the domain of $f(x) = \dfrac{x-1}{x^2-1}$

$$\lim_{x \to 1} \frac{x-1}{x^2-1} = \frac{1}{2}$$

(guessed from table)

$x < 1$	$f(x)$
0.5	0.666667
0.9	0.526316
0.99	0.502513
0.999	0.500250
0.9999	0.500025

$x > 1$	$f(x)$
1.5	0.400000
1.1	0.476190
1.01	0.497512
1.001	0.499750
1.0001	0.499975

$\downarrow$ 1 $\quad$ $\downarrow$ 0.5

$$a^2 - b^2 = (a-b)(a+b)$$

$$\lim_{x \to 1} \frac{x-1}{x^2-1} = \lim_{x \to 1} \frac{x-1}{x^2-1^2}$$

$$= \lim_{x \to 1} \frac{\cancel{(x-1)}}{\cancel{(x-1)}(x+1)}$$

$$= \lim_{x \to 1} \frac{1}{x+1} = \frac{1}{2}$$

Example 2 Estimate the value of $\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}$.

Note that 0 is not in the domain of $f(t) = \frac{\sqrt{t^2+9}-3}{t}$

.

t	$\frac{\sqrt{t^2+9}-3}{t^2}$
± 1.0	0.162277 . . .
± 0.5	0.165525 . . .
± 0.1	0.166620 . . .
± 0.05	0.166655 . . .
± 0.01	0.166666 . . . $\to \frac{1}{6}$

$$\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} = \frac{1}{6}$$

t	$\frac{\sqrt{t^2+9}-3}{t^2}$
± 0.001	0.166667
± 0.0001	0.166670
± 0.00001	0.167000
± 0.000001	0.000000

Calculators may lie!

$x \to a$ means x lies in $\underbrace{(a-h, a)}_{LHL} \cup \underbrace{(a, a+h)}_{RHL}$

One-sided limits

Right hand limit: When x approaches a from the right, that is, through values larger than a, the limit obtained is called right-hand limit and is written as

$$\lim_{x\to a^+} f(x) = L\,.$$

Left hand limit: When x approaches a from the left, that is, through values smaller than a, the limit obtained is called left-hand limit and is written as

$$\lim_{x\to a^-} f(x) = L\,.$$

$$\boxed{\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = L \ \text{ and } \ \lim_{x\to a^+} f(x) = L\,.}$$

when $\underset{x\to a}{RHL} \neq \underset{x\to a}{LHL}$, we say that limit does not exist.

we often write in this case $\lim_{x\to a} f(x) = \text{d.n.e.}$

Example 3.

The Heaviside function H is defined by

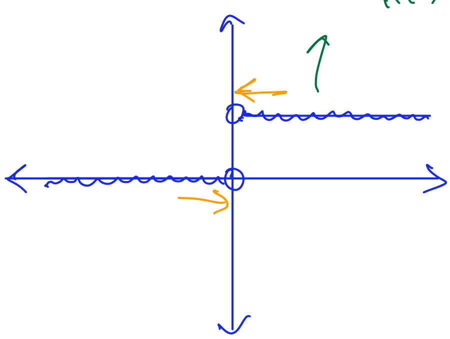
$$H(t) = \begin{cases} 0 & \text{if } t < 0\,, \\ 1 & \text{if } t > 0\,. \end{cases}$$

Guess the value of $\lim_{t\to 0} H(t)$.

$$\underset{t\to 0}{LHL}\, H(t) = \lim_{t\to 0^-} H(t) = \lim_{\underbrace{t\to 0^-}_{t<0}} 0 = 0$$

$$\underset{t\to 0}{RHL}\, H(t) = \lim_{t\to 0^+} H(t) = \lim_{\underbrace{t\to 0^+}_{t>0}} 1 = 1$$

Graph of $H(t)$

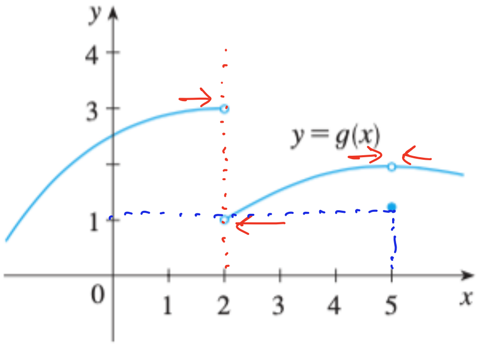


$$\underset{t\to 0}{LHL}\, H(t) \neq \underset{t\to 0}{RHL}\, H(t)$$

$\Rightarrow \lim_{t\to 0} H(t)$ does not exist.

Example 4.

The graph of a function g is shown below.



Use it to state the values:

1. $\lim\limits_{x\to 2^-} g(x)$. $\longrightarrow$ $= 3$
2. $\lim\limits_{x\to 2^+} g(x)$. $\longrightarrow$ $= 1$
3. $\lim\limits_{x\to 2} g(x)$. $\longrightarrow$ d.n.e.
4. $\lim\limits_{x\to 5^-} g(x)$. $\longrightarrow$ $= 2$
5. $\lim\limits_{x\to 5^+} g(x)$. $\longrightarrow$ $= 2$ $\Rightarrow g(5) = 1$
6. $\lim\limits_{x\to 5} g(x)$. $\longrightarrow$ $= 2$

Intuitive Definition of Infinite Limits Let f be a function defined on both sides of a except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a,

and

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a,

Example 5.

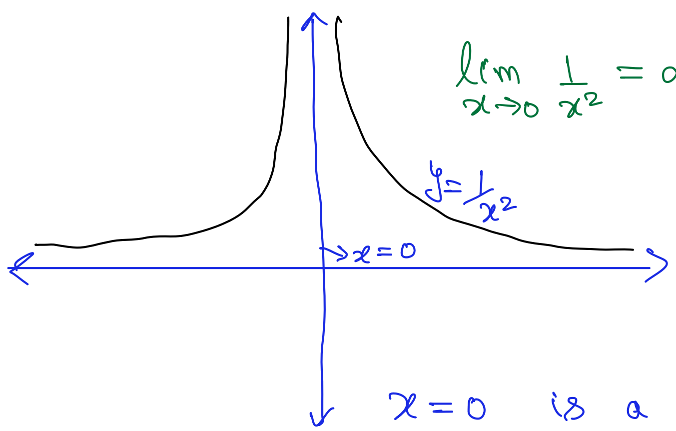
Find $\lim\limits_{x \to 0} \dfrac{1}{x^2}$ if it exists.

x	$f(x) = \frac{1}{x^2}$
± 0.1	$\frac{1}{0.1^2} = \frac{1}{0.01} = 100$
± 0.01	10000
± 0.001	1000000
$\vdots$	$\vdots$

$$\Rightarrow \lim_{x \to 0} \frac{1}{x^2} = \infty$$

$$\boxed{\frac{\text{finite}}{\to 0} = \pm\infty}$$

sign



$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

$x = 0$ is a vertical asymptote.

Vertical Asymptote

The vertical line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following statements is true:

1. $\lim_{x\to a} f(x) = \infty$
2. $\lim_{x\to a^-} f(x) = \infty$
3. $\lim_{x\to a^+} f(x) = \infty$
4. $\lim_{x\to a} f(x) = -\infty$
5. $\lim_{x\to a^-} f(x) = -\infty$
6. $\lim_{x\to a^+} f(x) = -\infty$

Example 6.

Find $\lim_{x\to 3^-} \frac{2x}{x-3}$, $\lim_{x\to 3^+} \frac{2x}{x-3}$ and $\lim_{x\to 3} \frac{2x}{x-3}$.

Is $x = 3$ a vertical asymptote of $f(x) = \dfrac{2x}{x-3}$?

$\underset{x\to 3}{\text{LHL}} \frac{2x}{x-3} = -\infty$, $\underset{x\to 3}{\text{RHL}} \frac{2x}{x-3} = \infty$

x	$f(x)$	x	$f(x)$
2.9	$\frac{2(2.9)}{-0.1} = -2(29)$	3.1	$2(31)$
2.99	$\frac{2(2.99)}{-0.01} = -2(299)$	3.01	$2(301)$
2.999	$\frac{2(2.999)}{-0.001} = -2(2999)$	3.001	$2(3001)$

$\lim_{x\to 3} \frac{2x}{x-3} = \text{d.n.e}$

Yes, $x=3$ is a vertical asympftote

$\lim_{x\to\infty}$ | $\lim_{x\to -\infty}$ → Chapter 3

Horizontal asymptotes