Learning objectives:

1. Compute limits using the limit laws.
2. Compute limits using the direct substitution property.
3. To be able to apply the squeeze theorem.

Limit Laws

Let $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then we have

1. $\lim_{x\to a}[f(x) \pm g(x)] = L \pm M$.
2. $\lim_{x\to a} f(x)g(x) = LM$.
3. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$ provided that $M \neq 0$.
4. $\lim_{x\to a} cf(x) = cL$.
5. $\lim_{x\to a} c = c$ where c is a constant.
6. $\lim_{x\to a} x = a$.
7. $\lim_{x\to a}[f(x)]^n = L^n$. $\longrightarrow$ $\lim_{x\to a} [f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$
8. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{L}$, given that $L \geq 0$ if n is even.

$\longrightarrow$ $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$

Example 1.

Evaluate the limit

$$\lim_{x\to 0} \frac{3x^2 + \sqrt{x} + 1}{2x^2 - x + 2} .$$

$$= \frac{\lim_{x\to 0} (3x^2 + \sqrt{x} + 1)}{\lim_{x\to 0} 2x^2 - x + 2} \quad \text{[Property 3]}$$

$$= \frac{\lim_{x\to 0} 3x^2 + \lim_{x\to 0} \sqrt{x} + \lim_{x\to 0} 1}{\lim_{x\to 0} 2x^2 - \lim_{x\to 0} x + \lim_{x\to 0} 2}$$

[Property 1]

[Properties 5,6,7,8]

$$= \frac{3\left(\lim_{x\to 0} x\right)^2 + \sqrt{\lim_{x\to 0} x} + 1}{2\left(\lim_{x\to 0} x\right)^2 - 0 + 2}$$

$$= \frac{3(0)^2 + \sqrt{0} + 1}{2(0)^2 - 0 + 2} = \frac{1}{2}$$

Direct substitution property

If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x\to a} f(x) = f(a)\,.$$

Example 2.

Evaluate the limit $\displaystyle\lim_{x\to 3}\frac{x^2-9}{x-3}$.

$\xrightarrow[\text{Substitution}]{\text{Direct}}$ $\dfrac{3^2-9}{3-3} = \dfrac{9-9}{3-3} = \boxed{\dfrac{0}{0}}$

indeterminate.

Factorize numerator and/or denominator.

$x^2-9 = x^2-3^2 = (x-3)(x+3)$

$\displaystyle\lim_{x\to 3}\frac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}} = \lim_{x\to 3}(x+3)$

Direct Substitution.

$= 3+3 = 6$

Example 3.

Evaluate $\displaystyle\lim_{h\to 0}\frac{(3+h)^2-9}{h}$. $\xrightarrow{\text{D.S.}}$ $\dfrac{(3+0)^2-9}{0} = \dfrac{0}{0}$

Factorize numerator (difference of squares) OR Expand the square in numerator.

$(3+h)^2 - 3^2 = [(3+h)-3]\,[(3+h)+3] = (\cancel{3}+h\cancel{-3})(3+h+3)$

$= h(6+h)$

$\displaystyle\lim_{h\to 0}\frac{\cancel{h}(6+h)}{\cancel{h}} = \lim_{h\to 0}(6+h) \underset{\text{D.S.}}{=} 6+0 = 6$

Example 4.

Find $\lim_{t\to 0} \dfrac{\sqrt{t^2+9}-3}{t^2}$. $\underset{=}{\text{DS}}$ $\dfrac{\sqrt{0+9}-3}{0^2} = \boxed{\dfrac{0}{0}}$

numerator is a difference of one/more radicals.

Rationalize the numerator.

$$\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} \times \frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3}$$

$$= \lim_{t\to 0} \frac{\overbrace{\left(\sqrt{t^2+9}-3\right)}^{\text{diff.}}\overbrace{\left(\sqrt{t^2+9}+3\right)}^{\text{sum}}}{t^2\left(\sqrt{t^2+9}+3\right)}$$ = diff. of squares.

$$= \lim_{t\to 0} \frac{\left(\sqrt{t^2+9}\right)^2 - 3^2}{t^2\left(\sqrt{t^2+9}+3\right)}$$

$$= \lim_{t\to 0} \frac{t^2 \cancel{+9} \cancel{-9}}{t^2\left(\sqrt{t^2+9}+3\right)} = \lim_{t\to 0} \frac{\cancel{t^2}}{\cancel{t^2}\left(\sqrt{t^2+9}+3\right)}$$

$$= \lim_{t\to 0} \frac{1}{\sqrt{t^2+9}+3} \stackrel{\text{DS}}{=} \frac{1}{\sqrt{0^2+9}+3}$$

$$= \frac{1}{3+3} = \frac{1}{6}$$

$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$ if and only if $\lim_{x\to a} f(x) = L$

Example 5.

Prove that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

compound function

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

whenever compound functions are involved check that LHL = RHL

LHL

$$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{\substack{x\to 0^- \\ (x<0)}} \frac{-x}{x} = \lim_{x\to 0^-} -1$$

$$= -1$$

RHL

$$\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{\substack{x\to 0^+ \\ (x>0)}} \frac{x}{x} = \lim_{x\to 0^+} 1 = 1$$

Since LHL $\neq$ RHL, $\lim_{x\to 0} \frac{|x|}{x}$ does not exist

Example 6.

If $f(x) = \begin{cases} \sqrt{x-4} & \text{if } x > 4\,, \\ 8-2x & \text{if } x < 4\,, \end{cases}$ then determine whether $\lim_{x\to 4} f(x)$ exists.

LHL

$$\lim_{x\to 4^-} f(x) = \lim_{\substack{x\to 4^- \\ (x<4)}} (8-2x) \overset{D.S.}{=} 8-2(4) = 0$$

RHL

$$\lim_{x\to 4^+} f(x) = \lim_{\substack{x\to 4^+ \\ (x>4)}} \sqrt{x-4} = \sqrt{\lim_{x\to 4^+} (x-4)}$$

$$\overset{D.S.}{=} \sqrt{4-4} = 0$$

LHL = RHL = 0

$$\Rightarrow \lim_{x\to 4} f(x) = 0$$

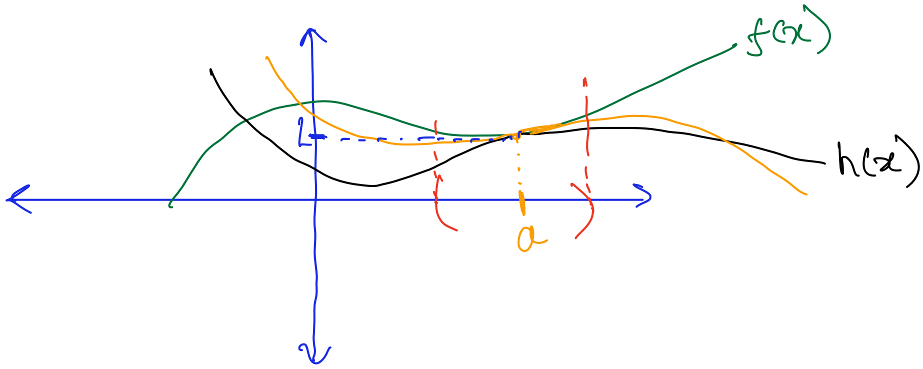
The Squeeze Theorem

If $f(x) \leq g(x) \leq h(x)$ when x is near a, except possibly at a itself, and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L\,,$$

then

$$\lim_{x \to a} g(x) = L\,.$$



Example 7.

Show that $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$.

$$\neq \lim_{x \to 0} x^2 \cdot \underbrace{\lim_{x \to 0} \sin\left(\frac{1}{x}\right)}_{\text{does not exist}}$$

because second limit does not exist

As $x \to 0$, $\frac{1}{x} \to \infty$ → sin function at ∞ oscillates between -1 and 1 and does not give a definite answer.

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

Multiply both sides of the two inequalities by x^2.

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$

the inequalities remain the same because x^2 is always +ve.

$$f(x) \leq g(x) \leq h(x)$$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} -x^2 = 0$, $\lim_{x \to 0} h(x) = \lim_{x \to 0} x^2 = 0$

$\Rightarrow \lim_{x \to 0} g(x) = \lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$ [using Squeeze Theorem]

Example 8.

Evaluate $\lim_{t\to 0}\left(\frac{1}{t} - \frac{1}{t^2+t}\right)$. $\stackrel{D.S.}{=} \left(\frac{1}{\to 0} - \frac{1}{\to 0}\right) = \boxed{\infty - \infty}$

indeterminate

$\downarrow$

Take common denominator and write as a single fraction

$$= \lim_{t\to 0} \frac{(t^2+t) - t}{t(t^2+t)}$$

$$= \lim_{t\to 0} \frac{t^2 \cancel{+t} \cancel{-t}}{t(t^2+t)} = \lim_{t\to 0} \frac{t^2}{t(t^2+t)} \stackrel{D.S.}{=} \boxed{\frac{0}{0}}$$

Factorize

$$= \lim_{t\to 0} \frac{t^2}{t[t(t+1)]} = \lim_{t\to 0} \frac{\cancel{t^2}}{\cancel{t^2}(t+1)}$$

$$= \lim_{t\to 0} \frac{1}{t+1} \stackrel{D.S.}{=} \frac{1}{0+1} = 1$$

Example 9.

Evaluate $\lim_{x\to 3} \frac{\frac{1}{x} - \frac{1}{3}}{x-3}$. $\stackrel{D.S.}{=} \frac{\frac{1}{3} - \frac{1}{3}}{3-3} = \boxed{\frac{0}{0}}$ Factorize

$$\lim_{x\to 3} \frac{\frac{1}{x} - \frac{1}{3}}{x-3} = \lim_{x\to 3} \frac{\frac{3-x}{3x}}{x-3}$$

$$= \lim_{x\to 3} \frac{3-x}{3x(x-3)} = \lim_{x\to 3} \frac{-\cancel{(x-3)}}{3x\cancel{(x-3)}}$$

$$= \lim_{x\to 3} \frac{-1}{3x} \stackrel{D.S.}{=} \frac{-1}{3(3)} = \frac{-1}{9}$$

Example 10.

Evaluate $\lim_{t \to 0} \dfrac{\sqrt{1+t} - \sqrt{1-t}}{t}$. $\underset{=}{\text{D.S.}}$ $\dfrac{\sqrt{1}-\sqrt{1}}{0} = \boxed{\dfrac{0}{0}}$

Numerator has radicals.
↓
Rationalize

$$= \lim_{t\to 0} \frac{\sqrt{1+t}-\sqrt{1-t}}{t} \cdot \frac{\sqrt{1+t}+\sqrt{1-t}}{\sqrt{1+t}+\sqrt{1-t}}$$

$$= \lim_{t\to 0} \frac{\left(\sqrt{1+t}-\sqrt{1-t}\right)\left(\sqrt{1+t}+\sqrt{1-t}\right)}{t\left(\sqrt{1+t}+\sqrt{1-t}\right)}$$

$$= \lim_{t\to 0} \frac{\left(\sqrt{1+t}\right)^2-\left(\sqrt{1-t}\right)^2}{t\left(\sqrt{1+t}+\sqrt{1-t}\right)}$$

$$= \lim_{t\to 0} \frac{(1+t)-(1-t)}{t\left(\sqrt{1+t}+\sqrt{1-t}\right)}$$

$$= \lim_{t\to 0} \frac{\cancel{1}+t\cancel{-1}+t}{t\left(\sqrt{1+t}+\sqrt{1-t}\right)}$$

$$= \lim_{t\to 0} \frac{2\cancel{t}}{\cancel{t}\left(\sqrt{1+t}+\sqrt{1-t}\right)}$$

$$= \lim_{t\to 0} \frac{2}{\sqrt{1+t}+\sqrt{1-t}} \underset{=}{\text{D.S.}} \frac{2}{\sqrt{1}+\sqrt{1}} = \frac{2}{2} = 1$$