Learning objectives:

1. Understand definition of continuity at a point.
2. Continuous functions on an interval.
3. Examples of continuous functions.
4. Continuity and composition of functions.
5. The intermediate value theorem and its applications.

Continuity at a point.

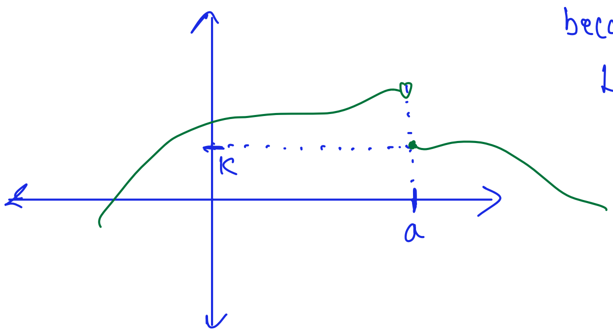
A function f is continuous at a number a if f is defined at a and

$$\lim_{x \to a} f(x) = f(a) .$$

If f is not continuous at a, then we say f is discontinuous at a.

Graphs of continuous functions.

If f is continuous at a then its graph cannot have a break at a.

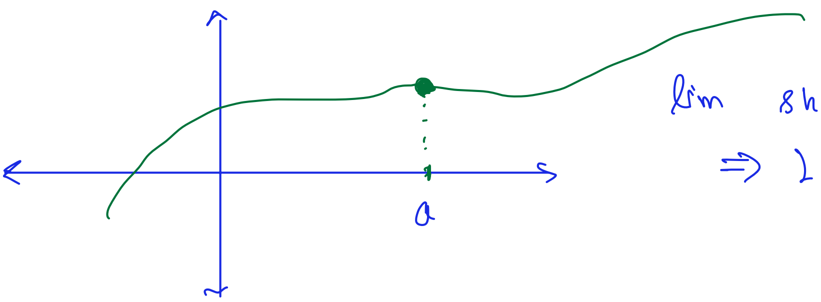


because of the break

LHL $\neq$ RHL at $x=a$

$\Rightarrow f(a) = K$

$\lim_{x \to a} f(x) =$ d.n.e.

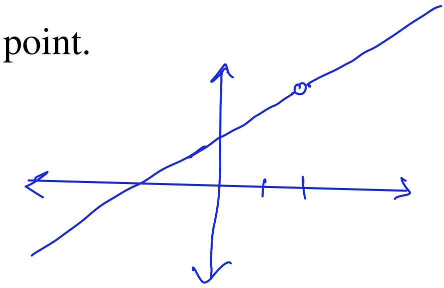


lim should exist

$\Rightarrow$ LHL $=$ RHL $=$ finite number.

Example 1.

Show that the following functions are discontinuous at the given point.

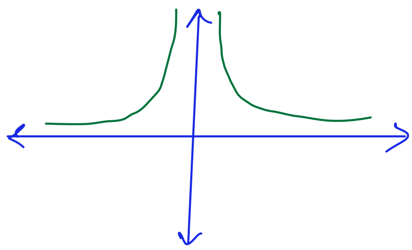


1. $f(x) = \dfrac{x^2 - x - 2}{x - 2}$ at $x = 2$.

$\rightarrow$ f should be defined at $x = 2$

But 2 is not in the domain of $f \Rightarrow f$ is discontinuous at $x = 2$.

2. $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0\,, \\ 1 & \text{if } x = 0\,, \end{cases}$ at $x = 0$.

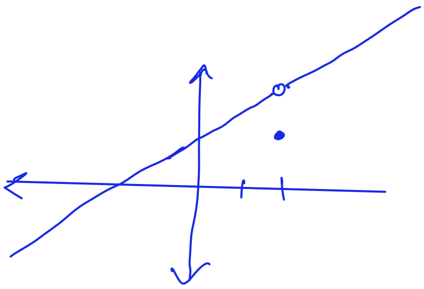


$f(0) = 1$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{x^2} = \infty$$

$\Rightarrow \lim_{x \to 0} f(x) \neq f(0) \Rightarrow f$ is discontinuous at $x = 0$

3. $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2\,, \\ 1 & \text{if } x = 2\,, \end{cases}$ at $x = 2$.



$\Rightarrow f(2) = 1$

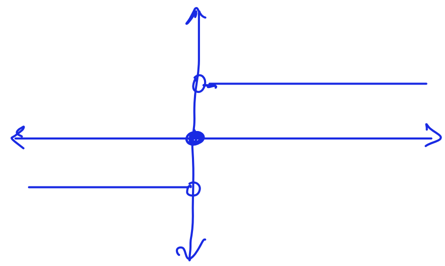
$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} \overset{D.S}{=} \frac{2^2 - 2 - 2}{2 - 2} = \boxed{\frac{0}{0}} \rightarrow \text{factorize.}$$

$$x^2 - x - 2 = \underbrace{x^2 - 2x}_{} + \underbrace{x - 2}_{} = \underbrace{x(x-2) + 1(x-2)}_{} = (x-2)(x+1)$$

$$\lim_{x \to 2} \frac{\cancel{(x-2)}(x+1)}{\cancel{(x-2)}} = \lim_{x \to 2} (x+1) \overset{D.S.}{=} 2 + 1 = 3$$

$\lim_{x \to 2} f(x) = 3 \neq 1 = f(2) \Rightarrow f$ is discontinuous at $x = 2$.

4. $f(x) = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0\,, \\ 0 & \text{if } x = 0\,, \end{cases}$ at $x = 0$.



$f(0)=0$

$$\underset{x=0}{\text{LHL}} = \lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = \lim_{x\to 0^-} -1 = -1$$

$$\underset{x=0}{\text{RHL}} = \lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} 1 = 1$$

$\text{LHL} \neq \underset{x=0}{\text{RHL}} \Rightarrow \lim_{x\to 0} f(x)$ does not exist $\Rightarrow f$ is discontinuous at $x=0$

Types of discontinuities

A discontinuity of f at a is called:

1. removable discontinuity if it can be removed by redefining f at $x = a$, Ex 1.1, 1.3
2. infinite discontinuity if the function takes an infinite (left hand and/or right hand) limit at $x = a$, Ex 1.2
3. jump discontinuity if both the left hand and right limits of the function at $x = a$ are finite but unequal. Ex 1.4

irremovable

In ① lim exists and is finite.

Continuous from the right and from the left

A function f is said to be continuous from the right at the number a if

$$\lim_{x\to a^+} f(x) = f(a)\,,$$

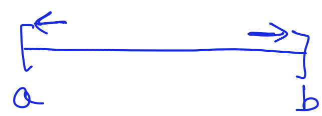
and f is said to be continuous from the left at a if

$$\lim_{x\to a^-} f(x) = f(a)\,.$$

Continuous on an interval

A function f is said to be continuous on an open interval (a, b) if it is continuous at every number in (a, b). $\rightarrow$ $a < x < b$

A function f is said to be continuous on a closed interval $[a, b]$ if it is continuous on (a, b), right continuous at a and left continuous at b.



Continuity on half-open intervals is defined similarly.

$[a, b)$, $(a, b]$

$\hookrightarrow$ right continuous at a $\quad\hookrightarrow$ left continuous at b

Example 2. Let $f(x) = \begin{cases} \dfrac{|x-1|}{x-1} & \text{if } x \neq 1\,, \\ 1 & \text{if } x = 1\,. \end{cases}$

Is f is continuous on the following intervals?

1. $[1, \infty)$.
2. $[0, 1]$.

Rewrite f : For $x \neq 1$, $f(x) = \dfrac{|x-1|}{x-1}$

If $x > 1$ then $x - 1 > 0$, so $|x-1| = x-1$

If $x < 1$ then $x - 1 < 0$, so $|x-1| = -(x-1)$

$$f(x) = \begin{cases} \dfrac{x-1}{x-1}, & x > 1 \\ \dfrac{-(x-1)}{x-1}, & x < 1 \\ 1, & x = 1 \end{cases} = \begin{cases} 1 & \text{if } x > 1 \\ -1 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

$$= \begin{cases} 1 & \text{if } x \geq 1 \\ -1 & \text{if } x < 1 \end{cases}$$

① Consider $[1, \infty)$

On the open interval $(1, \infty)$, $f(x) = 1$, a constant fns.

$\Rightarrow f$ is continuous on $(1, \infty)$ [no break in graph of f]

Is f right continuous at $x = 1$? $\longrightarrow$ Yes.

$$\text{RHL}_{x=1} = \lim_{x \to 1^+} f(x) = 1 = f(1)$$

$\Rightarrow f$ is continuous on $[1, \infty)$

② Consider $[0, 1]$. For $x < 1$, $f(x) = -1$ (a constant fn.)

$\Rightarrow f$ is continuous on $(-\infty, 1)$, hence also on $[0, 1)$

$\text{LHL}_{x=1} = \lim_{x \to 1^-} f(x) = -1 \neq f(1) = 1 \Rightarrow f$ is not left continuous at $x = 1$

$\Rightarrow f$ is not continuous on $[0,1]$

Combinations of continuous functions

If f and g are continuous at a and suppose c is a constant real number then the following functions are also continuous at a:

1. $f + g$,
2. $f - g$,
3. cf ,
4. fg ,
5. $\dfrac{f}{g}$, if $g(a) \neq 0$.

Examples of continuous functions Polynomials are continuous on $(-\infty, \infty)$

1. Polynomials are continuous everywhere, that is at every real number.
2. Rational functions are continuous in their domains.
3. Root functions are continuous in their domains.
4. Trigionometric functions are continuous in their domains. In particular, the $\underbrace{\text{sine and cosine functions}}_{\text{domain is } \mathbb{R}=(-\infty,\infty)}$ are continuous everywhere.

Example 3. On what intervals are the following functions continuous?

1. $f(x) = x^{1000} - 2x^{357} + 750$.
2. $g(x) = \dfrac{x^2 + x + 17}{x^2 - 1}$.
3. $h(x) = \sqrt{x} + \dfrac{x+1}{x-1} - \dfrac{x+1}{x-1}$.

① f is continuous on $(-\infty, \infty)$ and any subinterval of $(-\infty,\infty)$

② g is not defined when $x^2 - 1 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

g is continuous on $(-\infty,-1) \cup (-1,1) \cup (1,\infty)$ and any subinterval of that.

③ $x - 1 \neq 0 \Rightarrow x \neq 1$

$\sqrt{x} \rightsquigarrow$ defined only for $x \geq 0$

h is continuous on $[0,1) \cup (1,\infty)$ and any subintervals of that.

Example 4. Evaluate $\lim_{x\to\pi} \dfrac{\sin x}{\cos x + 2}$.

$\lim_{x \to a} f(x) = f(a)$

$-1 \le \cos x \le 1 \Rightarrow 2-1 \le 2+\cos x \le 2+1 \Rightarrow 1 \le \cos x + 2 \le 3$

$\Rightarrow$ In $\dfrac{\sin x}{\cos x + 2}$ denominator is never zero.

Therefore, $\dfrac{\sin x}{\cos x + 2}$ is continuous on all real number $(-\infty, \infty)$

$\lim_{x\to\pi} \dfrac{\sin x}{\cos x + 2} = \dfrac{\sin \pi}{\cos \pi + 2} = \dfrac{0}{-1+2} = \dfrac{0}{1} = 0$

Composition of continuous functions

If f is continuous at b and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x)) = f(b)$.

In other words, if f is continuous at $\lim_{x\to a} g(x)$, then

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) .$$

A consequence of the above statement is that if g is continuous at a and f is continuous at $g(a)$ then the composite function $f \circ g$ is continuous at a.

Example 5.

Where are the following functions continuous ?

1. $f(x) = \sin(x^2)$.
2. $g(x) = \dfrac{1}{\sqrt{x^2+7} - 4}$.

(1) x^2 is continuous on $(-\infty, \infty)$

$\sin x$ is continuous on $(-\infty, \infty)$, in particular on the range of $g(x) = x^2$

$\Rightarrow$ The composition $\sin(x^2)$ is continuous on $(-\infty, \infty)$

(2) numerator and denominator are both continuous on $(-\infty, \infty)$

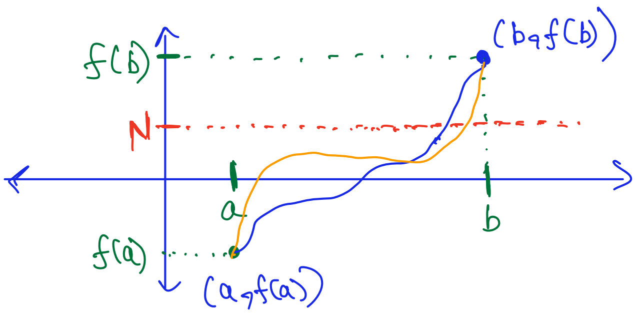
want to eliminate x for which denominator becomes 0.

$\sqrt{x^2+7} - 4 = 0 \Rightarrow \sqrt{x^2+7} = 4 \Rightarrow x^2+7 = 16 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$

$\Rightarrow g$ is continuous on $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$ / everywhere except at $3, -3$.

The intermediate value theorem.

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.



Example 6.

Show that there is a root of the equation

$$\underbrace{4x^3 - 6x^2 + 3x - 2}_{f(x)} = \underbrace{0}_{N}$$

between 1 and 2.

Let $f(x) = 4x^3 - 6x^2 + 3x - 2$ and $N = 0$

f is continuous on $[1, 2]$ $a=1$, $b=2$

$$f(1) = 4(1)^3 - 6(1)^2 + 3(1) - 2$$
$$= 4 - 6 + 3 - 2 = -1$$

$$f(2) = 4(2)^3 - 6(2)^2 + 3(2) - 2$$
$$= 32 - 24 + 6 - 2 = 12$$

<u>Note</u> $f(1) \neq f(2)$ and $f(1) < N < f(2)$

By IVT, there must be a number c such that $1 < c < 2$ and $f(c) = N = 0$

$\Rightarrow$ there must be a root between 1 and 2.