Learning objectives:

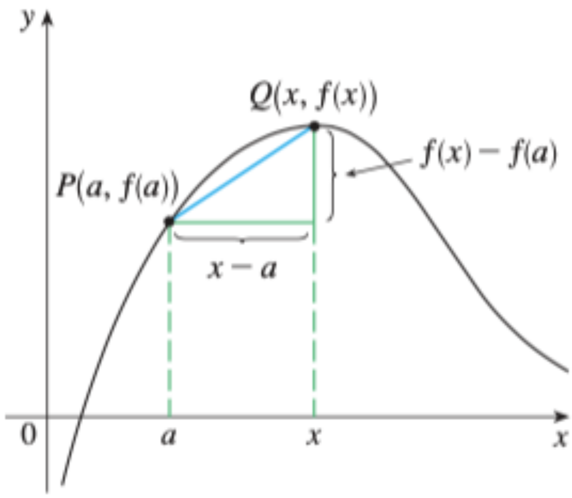
1. Using limits to find the slope of tangent line to a function at a given point.
2. Define the derivative of a function at a given point.
3. Interpret the derivative as an instantaneous rate of change of the dependent variable with respect to the independent variable.
4. Examples of rates of change: velocity and acceleration.

Slope of tangent line

The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

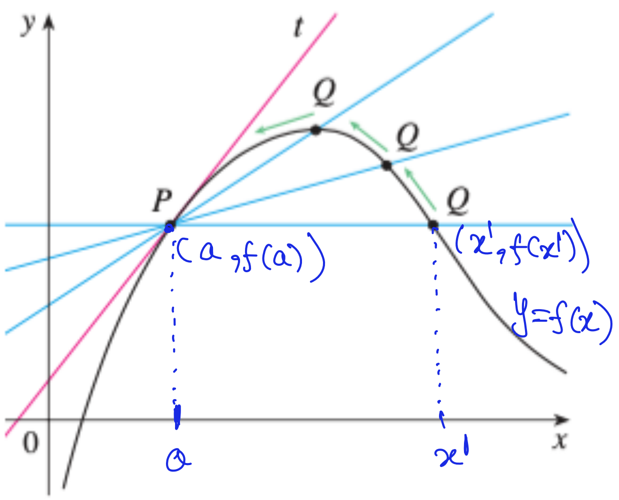
$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that the this limit exists.



$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

$$m_T = \lim_{Q \to P} m_{PQ}$$



$$= \lim_{x \to a} m_{PQ}$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$\parallel$

a number equal to slope of tangent at P

Example 1.

Find an equation of the tangent line to the hyperbola $y = 3/x$ at the point $P(3, 1)$.

a $f(a)$

$$m = \lim_{x\to 3} \frac{f(x) - f(3)}{x-3} \qquad a=3$$

$$= \lim_{x\to 3} \frac{\frac{3}{x} - \frac{3}{3}}{x-3} = \lim_{x\to 3} \frac{\frac{3}{x} - 1}{x-3} = \lim_{x\to 3} \frac{\frac{3-x}{x}}{x-3}$$

$$= \lim_{x\to 3} \frac{3-x}{x(x-3)} = \lim_{x\to 3} \frac{-1\cancel{(x-3)}}{x\cancel{(x-3)}} = \lim_{x\to 3} \frac{-1}{x} = \frac{-1}{3}$$

Slope-Point form for equation of a line :-

$$\frac{y-y_1}{x-x_1} = m \Rightarrow \frac{y-1}{x-3} = \frac{-1}{3} \Rightarrow 3(y-1) = -1(x-3)$$

$$\Rightarrow 3y - 3 = -x + 3 \Rightarrow x + 3y - 6 = 0$$

The derivative of a function at a point

$f(x) = \frac{3}{x}$

The derivative of a function f at a number a, denoted by $f'(a)$ is given by

we found

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x-a}$$

$f'(3) = -\frac{1}{3}$

if this limit exists.

If we write $x = a + h$, then we have $h = x - a$ so that $h \to 0$ as $x \to a$. Therefore,

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

Therefore, the slope of the tangent line to $y = f(x)$ at the point $(a, f(a))$ is given by $f'(a)$, the derivative of f at a.

Example 2.

Find the derivative of the function $f(x) = \sqrt{x}$ at the number a. $\to a \neq 0$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{\sqrt{a+h} - \sqrt{a}}{h} \underline{\underline{\text{DS}}} \frac{\sqrt{a} - \sqrt{a}}{0} = \boxed{\frac{0}{0}}$$

$$= \lim_{h\to 0} \frac{\sqrt{a+h} - \sqrt{a}}{h} \times \frac{\sqrt{a+h} + \sqrt{a}}{\sqrt{a+h} + \sqrt{a}}$$

Rationalize the numerator

$$= \lim_{h\to 0} \frac{(\sqrt{a+h} - \sqrt{a})(\sqrt{a+h} + \sqrt{a})}{h(\sqrt{a+h} + \sqrt{a})} = \lim_{h\to 0} \frac{(\sqrt{a+h})^2 - (\sqrt{a})^2}{h(\sqrt{a+h} + \sqrt{a})}$$

$$= \lim_{h\to 0} \frac{\cancel{a} + h - \cancel{a}}{h(\sqrt{a+h} + \sqrt{a})} = \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{a+h} + \sqrt{a})} \underline{\underline{\text{DS}}} \frac{1}{\sqrt{a} + \sqrt{a}} = \frac{1}{2\sqrt{a}}$$

Rates of Change

Let y depend on x via the function f, that is, $y = f(x)$.

If x changes from x_1 to x_2, the change (or increment) in x is $\Delta x = x_2 - x_1$.

The corresponding change in y is $\Delta y = f(x_2) - f(x_1)$.

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \qquad \text{(slope of the secant line PQ)}$$

is called the *average rate of change of y with respect to x over the interval* $[x_1, x_2]$.

Taking limit $\Delta x \to 0$, we obtain $\Delta x = h$, $x_1 = a$

$$\lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} \qquad \text{(slope of tangent line at P)}$$

the *instantaneous rate of change of y with respect to x at the instant* x_1. This is same as the derivative $f'(x_1)$.

$\boxed{\text{Thus, } f'(a) \text{ is the instantaneous rate of change of } y = f(x) \text{ w.r.t. } x \text{ at instant } a.}$

Examples of instantaneous rates of change

The velocity of a particle at a time instant t is the instantaneous rate of change of displacement of the particle with respect to time at t.

The acceleration of a particle at a time instant t is the instantaneous rate of change of velocity of the particle with respect to time at t.

Example 3. A particle moves along the x-axis with its displacement varying with time as $s(t) = t^2 - 3t + 1$. Find the velocity of the particle at the instant $t = 3$ seconds.

Velocity at $t=3$ s would be $s'(3)$

$$s'(3) = \lim_{h\to 0} \frac{s(3+h) - s(3)}{h}$$

$$= \lim_{h\to 0} \frac{(3+h)^2 - 3(3+h) + 1 - (3^2 - 3(3) + 1)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{3^2} + h^2 + 2(3)(h) - \cancel{9} - 3h + \cancel{1} - \cancel{1}}{h}$$

$$= \lim_{h\to 0} \frac{h^2 + 6h - 3h}{h}$$

$$= \lim_{h\to 0} \frac{h^2 + 3h}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(h+3)}{\cancel{h}} = \lim_{h\to 0} h+3 = 3$$

$\Rightarrow$ velocity at $t=3$ is 3 m/s.

Example 4. A particle is moving along a straight line with its velocity varying with time as $v(t) = (t^2 + 1)/t$. Find the acceleration of the particle at $t = 1$ seconds.

acceleration at $t=1s$ would be $v'(1)$

$$v'(1) = \lim_{h\to 0} \frac{v(1+h) - v(1)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{(1+h)^2+1}{1+h} - \frac{1^2+1}{1}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{(1+h)^2+1}{1+h} - 2}{h}$$

$$= \lim_{h\to 0} \frac{\frac{(1+h)^2+1-2(1+h)}{1+h}}{h}$$

$$= \lim_{h\to 0} \frac{(1+h)^2+1-2(1+h)}{h(1+h)}$$

$$= \lim_{h\to 0} \frac{\cancel{1}+h^2+\cancel{2h}+\cancel{1}-\cancel{2}-\cancel{2h}}{h(1+h)}$$

$$= \lim_{h\to 0} \frac{h^{\cancel{2}}}{\cancel{h}(1+h)} = \lim_{h\to 0} \frac{h}{1+h} = 0 \text{ m/s}^2$$