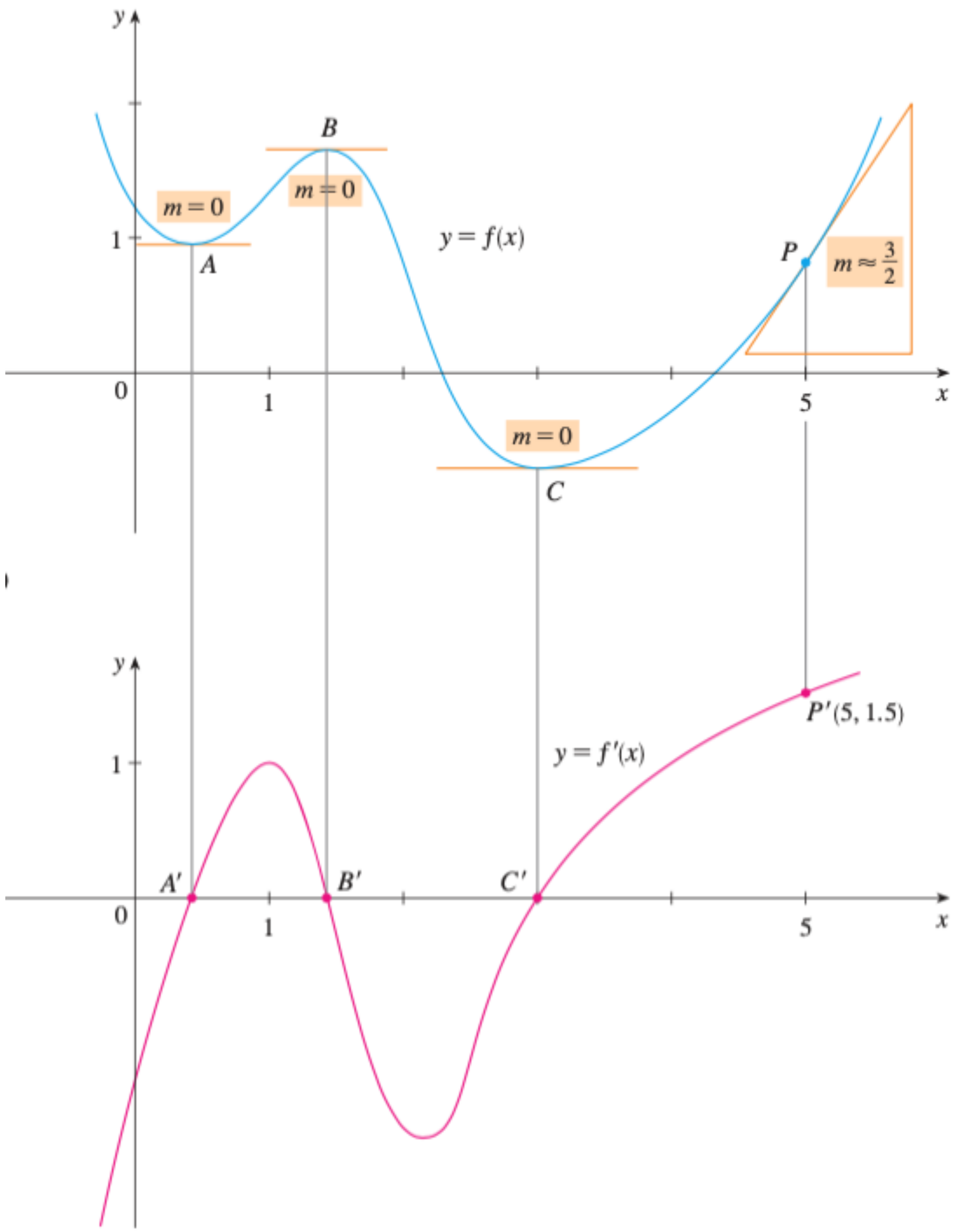
Learning objectives:

1. Define the derivative as a function.
2. The property of differentiability
3. When can a function fail to be differentiable?
4. Higher derivatives and their interpretation.

The derivative of a function $y = f(x)$ is a new function $f'(x)$ defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$



Example 1. If $f(x) = x^3 - x$, find a formula for $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left[(x+h)^3 - (x+h)\right] - \left[x^3 - x\right]}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^3} + h^3 + 3x^2h + 3xh^2 - \cancel{x} - h - \cancel{x^3} + \cancel{x}}{h}$$

$$= \lim_{h \to 0} \frac{h^3 + 3x^2h + 3xh^2 - h}{h} = \lim_{h \to 0} \frac{\cancel{h}(h^2 + 3x^2 + 3xh - 1)}{\cancel{h}}$$

$$= \lim_{h \to 0} h^2 + 3x^2 + 3xh - 1 = 3x^2 - 1$$

Example 2. Find $f'(x)$ if $f(x) = \dfrac{1-x}{2+x}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1-(x+h)}{2+(x+h)} - \frac{1-x}{2+x}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1-x-h}{2+x+h} - \frac{1-x}{2+x}}{h} = \lim_{h \to 0} \frac{\frac{(1-x-h)(2+x) - (1-x)(2+x+h)}{(2+x+h)(2+x)}}{h}$$

$$= \lim_{h \to 0} \frac{(1-x-h)(2+x) - (1-x)(2+x+h)}{h(2+x+h)(2+x)}$$

$$= \lim_{h \to 0} \frac{2(1-x-h) + x(1-x-h) - (2+x+h) - (-x)(2+x+h)}{h(2+x+h)(2+x)}$$

$$= \lim_{h \to 0} \frac{\cancel{2} - \cancel{2x} - 2h + \cancel{x} - \cancel{x^2} - \cancel{xh} - \cancel{2} - \cancel{x} - h + \cancel{2x} + \cancel{x^2} + \cancel{xh}}{h(2+x+h)(2+x)}$$

$= \lim \; -3\cancel{h}$ D.S. -3 $\leftarrow f'(x)$

$h \to 0 \quad \cancel{h}(a+x+h)(a+x) \qquad (a+x)^2$

Other Notations for Derivative

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)\,.$$

The symbol D and d/dx are called the differentiation operators since they indicate the process of differentiation.

We often write $f'(a)$ as $\left.\frac{dy}{dx}\right|_{x=a}$.

Differentiability

A function f is said to be differentiable at a if $f'(a)$ exists. It is differentiable on an open interval if it is differentiable at every number in the interval.

Example of $y = \sqrt{x}$ from previous lecture: $f'(a) = \frac{1}{2\sqrt{a}}$ $\Rightarrow y=\sqrt{x}$ function is not differentiable at $x=0$

$Df = [0, \infty)$

$Df' = (0, \infty)$

Example 3. Where is function $f(x) = |x|$ differentiable?

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

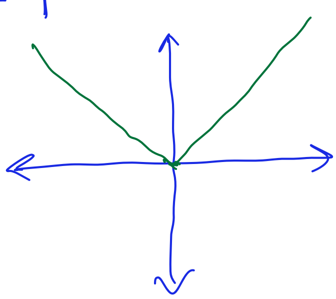
$\underline{x > 0}$ $\longrightarrow$ can take small h so that $x+h > 0$

$$f'(x) = \lim_{h \to 0} \frac{|x+h| - |x|}{h} = \lim_{h \to 0} \frac{\cancel{x}+h - \cancel{x}}{h} = 1$$

$\underline{x < 0}$

$$f'(x) = \lim_{h \to 0} \frac{|x+h| - |x|}{h} = \lim_{h \to 0} \frac{-(x+h) - (-x)}{h}$$

$$= \lim_{h \to 0} \frac{-\cancel{x} - h + \cancel{x}}{h} = -1$$



$\underline{x = 0}$ $\quad f'(0) = \lim_{h \to 0} \frac{|h| - 0}{h} = \lim_{h \to 0} \frac{|h|}{h}$ $\quad f(0) = 0$

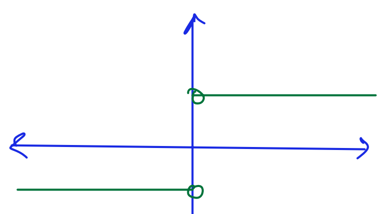
LHL $\to -1$

RHL $\to +1$

$\Rightarrow$ limit d.n.e.

$\Rightarrow |x|$ is not differentiable at $x = 0$

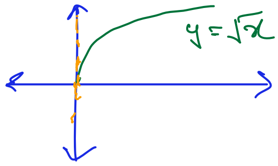
$$f'(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$



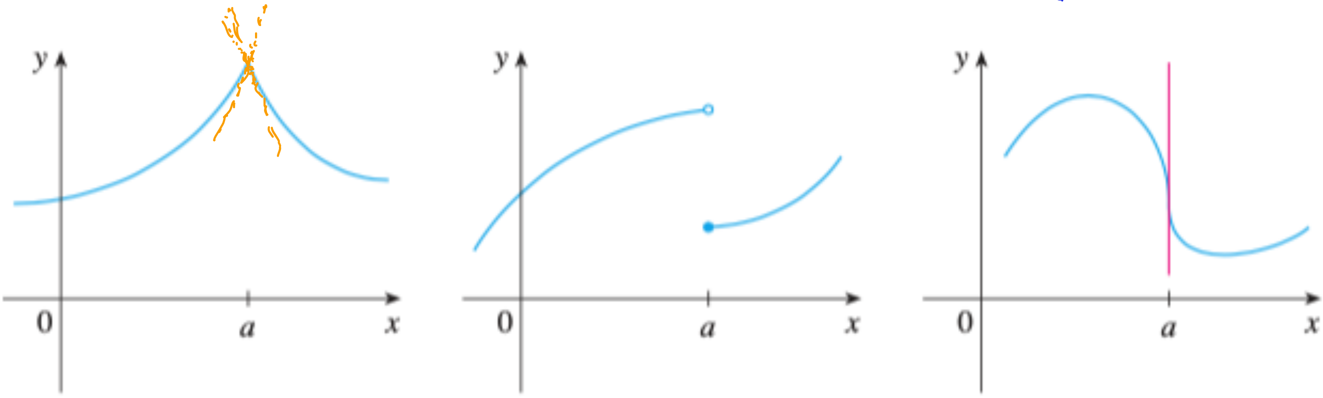
Differentiability implies continuity

If f is differentiable at a then f is continuous at a. $\Rightarrow$ discontinuity implies not differentiable.

There exist functions that are continuous but not differentiable.



How can a function fail to be differentiable?



(a) A corner (b) A discontinuity (c) A vertical tangent

Higher Derivatives

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}.$$ ← second derivative

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$ ↑ third derivative

Same as $f'' = (f')'$ we have $f^{(n)} = (f^{(n-1)})'$, that is, in general

$f''' = (f'')'$

$$\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n}.$$ ← n^{th} derivative

↑ $(n-1)^{th}$ derivative

Position (function) $\xrightarrow{\text{derivative}}$ velocity $\xrightarrow{\text{derivative}}$ acceleration $\xrightarrow{\text{derivative}}$ jerk

↑ first derivative. ↑ second derivative ↑ third derivative

$s(t) = t^3 - t^2$

Position function.

$f^{(4)} = (f''')'$

Example 4. If $f(x) = x^3 - x$, find $f''(x)$, $f'''(x)$ and $f^{(4)}(x)$.

In Example 1, $f'(x) = 3x^2 - 1$

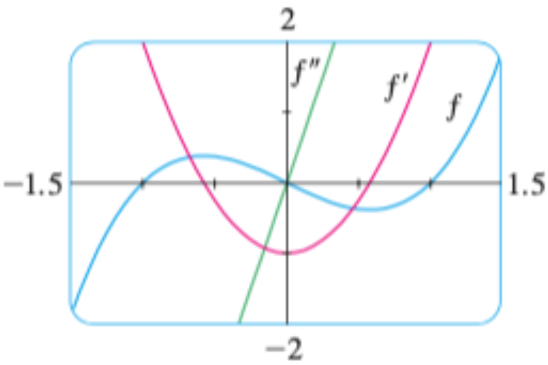
$$f''(x) = [f'(x)]' = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h}$$

$$= \lim_{h\to 0} \frac{[3(x+h)^2 - 1] - [3x^2 - 1]}{h}$$

$$= \lim_{h\to 0} \frac{3(x^2 + h^2 + 2xh) \cancel{-1} - 3x^2 \cancel{+1}}{h} = \lim_{h\to 0} \frac{\cancel{3x^2} + 3h^2 + 6xh \cancel{-3x^2}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(3h + 6x)}{\cancel{h}} = \lim_{h\to 0} (3h + 6x) = 6x$$

$$\Rightarrow f''(x) = 6x$$



$$f'''(x) = \lim_{h\to 0} \frac{f''(x+h) - f''(x)}{h} = \lim_{h\to 0} \frac{6(x+h) - 6x}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{6x} + 6h \cancel{-6x}}{h} = 6 \Rightarrow f'''(x) = 6$$

$$f^{(4)}(x) = \lim_{h\to 0} \frac{f'''(x+h) - f'''(x)}{h} = \lim_{h\to 0} \frac{6-6}{h} = 0$$

$$\frac{0}{\to 0}$$

$$\Rightarrow f^{(4)}(x) = 0$$