Learning objectives:

1. Derivatives of power functions: The power rule
2. Taking derivatives of combinations of functions: the constant multiple rule, the sum and difference rules, the product rule, the quotient rule

Derivative of a constant function

$$\frac{d}{dx}(c) = 0\,.$$

Derivative of a power function

$$\frac{d}{dx}(x^n) = nx^{n-1}\,.$$

$$[x^n]' = n\,x^{n-1}$$

Here n can be any real number.

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}}$$

Example 1. Find derivative of the following functions.

1. $f(x) = x^{600}$.
2. $f(x) = 1/x^2$.
3. $f(x) = x^{\pi}$.
4. $f(x) = \sqrt[100]{x^3}$.

① $f'(x) = 600\,x^{599}$

② $f'(x) = \frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{d}{dx}\left(x^{-2}\right) = -2\,x^{-2-1} = -2x^{-3} = \frac{-2}{x^3}$

③ $f'(x) = \pi\,x^{\pi-1}$

$$\frac{d}{dx}\left(x^{\sqrt{2}}\right) = \sqrt{2}\,x^{\sqrt{2}-1}$$

$$\frac{d}{dx}\left(x^{1.414}\right) = 1.414\,x^{0.414}$$

④ $f'(x) = \frac{d}{dx}\left(\left(x^3\right)^{\frac{1}{100}}\right)$

$$= \frac{d}{dx}\left(x^{\frac{3}{100}}\right) = \frac{3}{100}\,x^{\frac{3}{100}-1} = \frac{3}{100}\,x^{\frac{-97}{100}}$$

$$= \frac{3}{100\,x^{97/100}} = \frac{3}{100\,\sqrt[100]{x^{97}}}$$

The constant multiple rule

$$\frac{d}{dx}\big(cf(x)\big) = c\frac{d}{dx}f(x)\,.$$

Example 2. Find derivative of the following functions.

1. $f(x) = 3x^4$.
2. $f(x) = -x$.

① $f'(x) = \frac{d}{dx}(3x^4) = 3\frac{d}{dx}(x^4) = 3(4x^3) = 12x^3$

② $f'(x) = \frac{d}{dx}(-x) = -1\frac{d}{dx}(x) = -1(1) = -1$

The sum rule

$$\frac{d}{dx}\big(f(x) + g(x)\big) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)\,.$$

The difference rule

$$\frac{d}{dx}\big(f(x) - g(x)\big) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)\,.$$

Example 3. Find the derivative of $f(x) = x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5$.

$f'(x) = (x^8)' + (12x^5)' - (4x^4)' + (10x^3)' - (6x)' + (5)'$

(Sum and difference rule)

$f'(x) = 8x^7 + 12(5x^4) - 4(4x^3) + 10(3x^2) - 6(1) + 0$

$= 8x^7 + 60x^4 - 16x^3 + 30x^2 - 6$

Example 4. Find the points of the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

Find values of x for which $\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = 4x^3 - 12x$$

$$4x^3 - 12x = 0 \Rightarrow 4x(x^2 - 3) = 0$$

$$\Rightarrow x = 0 \text{ or } x^2 - 3 = 0 \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$$

$x = 0, -\sqrt{3}, +\sqrt{3}$ are reqd. points.

Example 5. The position function of a particle is $s(t) = 2t^3 - 5t^2 + 3t + 4$ where s is measured in meters and t is measured in seconds. Find the time instants where the particle is at rest. Find the the acceleration as a function of time. What is the acceleration after 2 seconds?

$$\Rightarrow v(t) = s'(t) = 6t^2 - 10t + 3$$

$$6t^2 - 10t + 3 = 0 \Rightarrow t = \frac{10 \pm \sqrt{100 - 4(6)(3)}}{12}$$

$$= \frac{10 \pm \sqrt{28}}{12} = \frac{10 \pm 2\sqrt{7}}{12} = \frac{5 \pm \sqrt{7}}{6} \text{ s.}$$

time instants of rest.

$$a(t) = \frac{d}{dt}(6t^2 - 10t + 3) = 12t - 10$$

$$a(2) = 12(2) - 10 = 14 \text{ m/s}^2$$

The product rule

$$= f(x)\,g'(x) + f'(x)\,g(x)$$

$$\frac{d}{dx}\big(f(x)g(x)\big) = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)\,.$$

$$\neq f'(x)\,g'(x)$$

Example 6 : Differentiate the function $f(t) = \sqrt{t}\,(a+bt)$

Alternatively $a\sqrt{t} + bt\sqrt{t}$ $\parallel$ $a t^{1/2} + b t^{3/2}$

$$\Rightarrow f'(t) = \frac{d}{dt}\left(\sqrt{t}\,(a+bt)\right)$$

$$= \sqrt{t}\,\frac{d}{dt}(a+bt) + (a+bt)\frac{d}{dt}(\sqrt{t})$$

$$\frac{d}{dt}\left(t^{1/2}\right) = \frac{1}{2}t^{1/2-1}$$

$$= \sqrt{t}\,[0+b] + (a+bt)\frac{1}{2\sqrt{t}}$$

$$= b\sqrt{t} + \frac{a}{2\sqrt{t}} + \frac{bt}{2\sqrt{t}} = b\sqrt{t} + \frac{a}{2\sqrt{t}} + \frac{b}{2}\sqrt{t}$$

$$= \boxed{\frac{a}{2\sqrt{t}} + \frac{3b}{2}\sqrt{t}} = \boxed{\frac{a+3bt}{2\sqrt{t}}}$$

Example 7 : If $h(x) = x\,g(x)$ and $g(3) = 5$, $g'(3) = 2$. Find $h'(3)$.

$$h'(x) = \left[\frac{d}{dx}(x)\right] g(x) + x\left[\frac{d}{dx}g(x)\right]$$

$$\Rightarrow h'(x) = g(x) + x\,g'(x) \quad \text{[By Product rule]}$$

$$\Rightarrow h'(3) = g(3) + 3\,g'(3) = 5 + 3(2) = \boxed{11}$$

The quotient rule

requires Chain rule (2.5)

$\sqrt{x+2}$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{\left(g(x)\right)^2}.$$

Example 8 Let $y = \dfrac{x^2+x-2}{x^3+6}$. Find $\dfrac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{(x^3+6)(x^2+x-2)' - (x^2+x-2)(x^3+6)'}{(x^3+6)^2}$$

$$= \frac{(x^3+6)(2x+1) - (x^2+x-2)\,3x^2}{(x^3+6)^2}$$

$$= \frac{2x^4+x^3+12x+6-3x^4-3x^3+6x^2}{(x^3+6)^2}$$

$$= \frac{-x^4-2x^3+6x^2+12x+6}{(x^3+6)^2}$$

Example 9 : Find equations of tangent line and normal line to the curve $y = \frac{\sqrt{x}}{1+x^2}$ at the point $(1, \frac{1}{2})$.

$$\frac{dy}{dx} = \frac{(1+x^2)[\sqrt{x}]' - \sqrt{x}\,[1+x^2]'}{(1+x^2)^2}$$

$$= \frac{(1+x^2)\frac{1}{2\sqrt{x}} - \sqrt{x}\,(2x)}{(1+x^2)^2} = \frac{\frac{(1+x^2) - 2\sqrt{x}\cdot\sqrt{x}\,(2x)}{2\sqrt{x}}}{(1+x^2)^2}$$

$$= \frac{(1+x^2) - 2\sqrt{x}\cdot\sqrt{x}\,(2x)}{2\sqrt{x}\,(1+x^2)^2} = \frac{1+x^2 - 4x^2}{2\sqrt{x}\,(1+x^2)^2}$$

$$= \frac{1-3x^2}{2\sqrt{x}\,(1+x^2)^2}$$

Need slope of tangent at $x=1$

$$m_T = \left.\frac{dy}{dx}\right|_{x=1} = \frac{1-3(1)^2}{2\sqrt{1}\,(1+1^2)^2} = \frac{-2}{2\,(2)^2} = \frac{-1}{4}$$

Tangent line at $(1, \frac{1}{2})$

$$\frac{y-\frac{1}{2}}{x-1} = \frac{-1}{4} \Rightarrow 4\left(y-\frac{1}{2}\right) = -1\,(x-1)$$

$$\Rightarrow 4y-2 = -x+1 \Rightarrow x+4y-2-1=0$$

$$\Rightarrow x+4y-3=0$$

Normal line : the line perpendicular to the tangent line.

$$m_T\, m_N = -1 \Rightarrow \frac{-1}{4}\cdot m_N = -1 \Rightarrow m_N = 4$$

Normal line at $(1, \frac{1}{2})$

$$\frac{y-\frac{1}{2}}{x-1} = 4 \Rightarrow y-\frac{1}{2} = 4(x-1) \Rightarrow y-\frac{1}{2} = 4x-4 \Rightarrow 4x-y-4+\frac{1}{2}=0$$

$$\Rightarrow 4x-y-\frac{7}{2}=0$$

$$\Rightarrow 8x-2y-7=0$$

Example 10 : At what points on the hyperbola $xy=12$ is the tangent line parallel to the line $3x+y=0$?

Diff. $\longrightarrow$ $\frac{dy}{dx}$ in terms of x = slope of $(3x+y=0)$

$\Downarrow$

$3x+y=0 \Rightarrow y=\underbrace{-3}_{\text{slope of this line}}x$

$m=-3$

$\Rightarrow \frac{dy}{dx} = -3$

solve for x.

$xy=12 \Rightarrow y=\frac{12}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\frac{12}{x}\right) = 12\frac{d}{dx}\left(\frac{1}{x}\right) = 12\frac{d}{dx}\left(x^{-1}\right) = 12(-1)x^{-2}$

$= \frac{-12}{x^2}$

$\Rightarrow \frac{-12}{x^2} = -3$

Solve for x

$\Rightarrow -12 = -3x^2 \Rightarrow x^2 = \frac{-12}{-3} = 4$

$\Rightarrow x^2 = 4$

$\Rightarrow x = \pm\sqrt{4} \Rightarrow x=\pm 2$

The tangent is parallel to $y+3x=0$ at $x=2$ and $x=-2$.