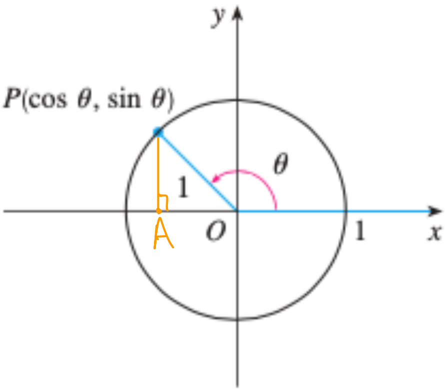
Learning objectives:

1. Derivatives of trigonometric functions.
2. Derivatives of combinations of trigonometric functions.

Trigonometric Functions

$\sin\theta$ is the length of perpendicular and $\cos\theta$ is the length of base in the triangle formed OPA inside a unit circle.



$\sin\theta = PA$, $\cos\theta = OA$ (measured with sign)

$0 \quad 30° \quad 45° \quad 60° \quad 90°$

x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin x$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1
$\cos x$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0

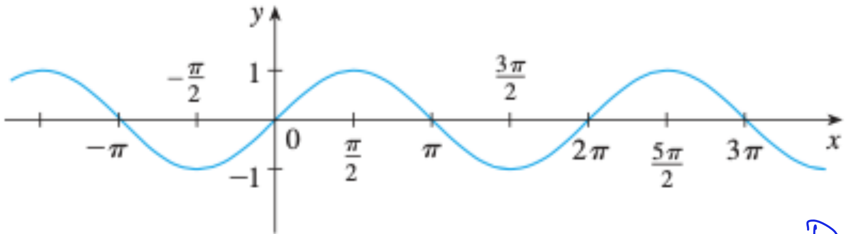
$i = 0 \quad 1 \quad 2 \quad 3 \quad 4$

$\sin x = \sqrt{\frac{i}{4}}$

We have

$\sin(\pi - x) = \sin x\,, \quad \sin(\pi + x) = -\sin x\,, \quad \sin(2\pi - x) = -\sin x\,, \quad \sin(-x) = -\sin x\,.$ odd function

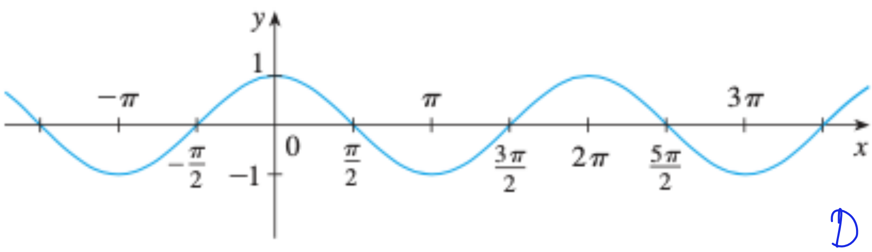
$\underbrace{\cos(\pi - x) = -\cos x}_{2^{nd}\text{ quadrant}}\,, \quad \underbrace{\cos(\pi + x) = -\cos x}_{3^{rd}\text{ quadrant}}\,, \quad \underbrace{\cos(2\pi - x) = \cos x}_{4^{th}\text{ quadrant}}\,, \quad \cos(-x) = \cos x\,.$ even function



(a) $f(x) = \sin x$

$D(\sin x) = \mathbb{R}$
Domain

$\text{Range}(\sin x) = [-1, 1]$



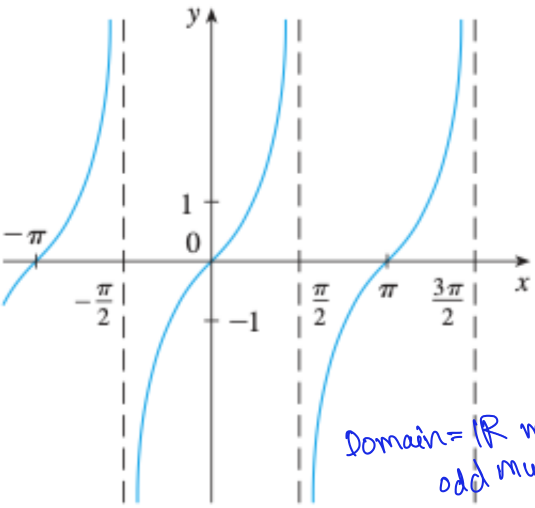
(b) $g(x) = \cos x$

$D(\cos x) = \mathbb{R}$
Domain

$\text{Range}(\cos x) = [-1, 1]$

The other trigonometric functions are defined as:

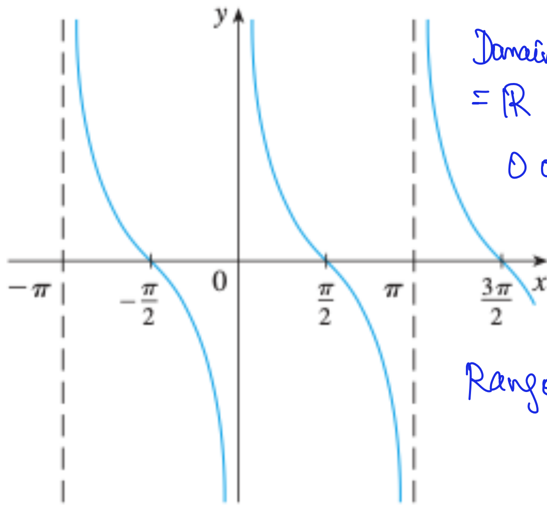
$$\tan x = \frac{\sin x}{\cos x}\,, \quad \cot x = \frac{\cos x}{\sin x}\,, \quad \sec x = \frac{1}{\cos x}\,, \quad \csc x = \frac{1}{\sin x}\,.$$



(a) $y = \tan x$

Domain $= \mathbb{R}$ minus odd multiples of $\pm\pi/2$

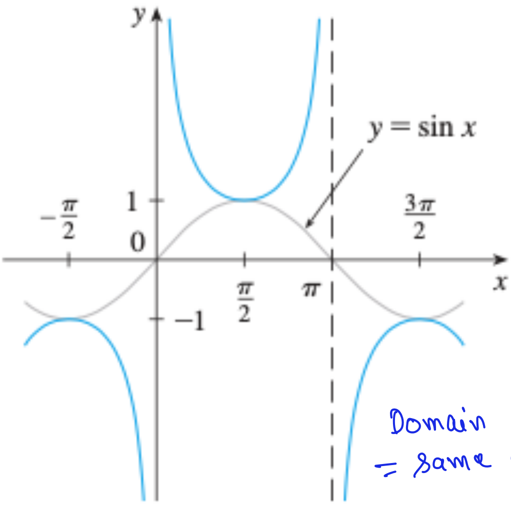
Range $= \mathbb{R}$



(b) $y = \cot x$

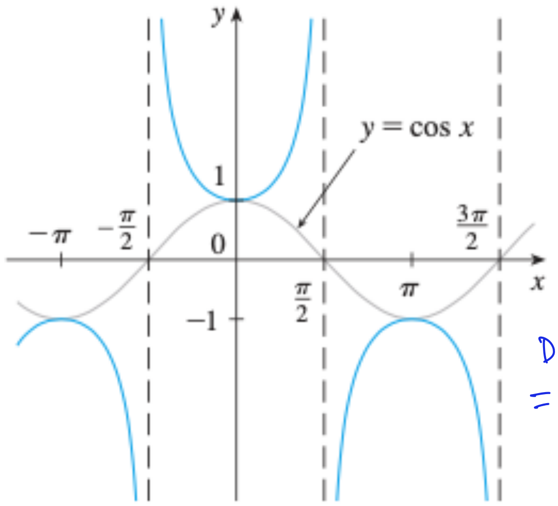
Domain $= \mathbb{R}$ minus 0 or multiples of $\pm\pi$

Range $= \mathbb{R}$



(c) $y = \csc x$

Domain = same as $\cot x$



(d) $y = \sec x$

Domain = same as $\tan x$

Range of $\sec x \,|\, \csc x = (-\infty, -1] \cup [1, \infty)$

Trigonometric Identities

$$\left[\sin^2 x + \cos^2 x = 1\,, \quad 1 + \tan^2 x = \sec^2 x\,, \quad 1 + \cot^2 x = \csc^2 x\,.\right] \rightarrow$$ Pythagoras theorem.

$$\left[\sin(x+y) = \sin x \cos y + \cos x \sin y\,, \quad \cos(x+y) = \cos x \cos y - \sin x \sin y\,.\right]$$ derivative

$$\left[\sin x \cos x = \frac{1}{2}\sin 2x\,, \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x)\,, \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x)\,.\right] \rightarrow$$ Integrals

Two important limits

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\,,\qquad \lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}=0\,.$$

Example 1. Evaluate the following limits:

1. $\lim_{x\to 0}\dfrac{\sin 7x}{4x}$.
2. $\lim_{x\to 0}\dfrac{\sin 10x}{\sin 5x}$.

(1) $$\lim_{x\to 0}\frac{\sin 7x}{4x}=\frac{1}{4}\lim_{x\to 0}\frac{\sin 7x}{x}=\frac{1}{4}\lim_{y\to 0}\frac{\sin y}{y/7}$$

As $x\to 0$, $y=7x\to 0$

$\Rightarrow \frac{y}{7}=x$

$$=\frac{1}{4}\lim_{y\to 0}7\,\frac{\sin y}{y}$$

$$=\frac{7}{4}\underbrace{\lim_{y\to 0}\frac{\sin y}{y}}_{1}=\frac{7}{4}$$

(2) $$\lim_{x\to 0}\frac{\sin 10x}{\sin 5x}$$

$$=\lim_{x\to 0}\frac{10x\,\dfrac{\sin 10x}{10x}}{5x\,\dfrac{\sin 5x}{5x}}=\lim_{x\to 0}\frac{10\cancel{x}}{5\cancel{x}}\;\frac{\lim_{x\to 0}\dfrac{\sin 10x}{10x}}{\lim_{x\to 0}\dfrac{\sin 5x}{5x}}=2\,\frac{\lim_{y\to 0}\dfrac{\sin y}{y}}{\lim_{z\to 0}\dfrac{\sin z}{z}}$$

Put $y=10x$ in numerator
Put $z=5x$ in denomitor
As $x\to 0$, $y\to 0$ and $z\to 0$

$$=2$$

Example 2. Evaluate $\lim_{x\to 0} x\cot x$.

$$\lim_{x\to 0} x\cot x \xrightarrow{\text{D.S.}} \underbrace{0\times(\pm\infty)}_{\text{Indeterminate.}}$$

$$\|$$

$$\lim_{x\to 0} x\,\frac{\cos x}{\sin x}=\underbrace{\lim_{x\to 0}\cos x}_{\cos 0=1}\;\lim_{x\to 0}\frac{x}{\sin x}=\lim_{x\to 0}\frac{x}{\sin x}$$

$$=\lim_{x\to 0}\frac{\cancel{x}}{\cancel{x}\cdot\dfrac{\sin x}{x}}=\frac{1}{\lim_{x\to 0}\dfrac{\sin x}{x}}=1$$

Derivatives of trigonometric functions

$$\frac{d}{dx}(\sin x) = \cos x\,, \qquad \frac{d}{dx}(\cos x) = -\sin x\,,$$

$$\frac{d}{dx}(\tan x) = \sec^2 x\,, \qquad \frac{d}{dx}(\cot x) = -\csc^2 x\,,$$

$$\frac{d}{dx}(\sec x) = \sec x\,\tan x\,, \qquad \frac{d}{dx}(\csc x) = -\csc x\,\cot x\,.$$

1 $$\frac{d}{dx}(\sin x) = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

$$= \sin x\,\underbrace{\lim_{h\to 0}\frac{\cos h - 1}{h}}_{0} + \cos x\,\underbrace{\lim_{h\to 0}\frac{\sin h}{h}}_{1} = \cos x$$

2 $\frac{d}{dx}(\cos x) \to$ similare use $\cos(x+h) = \cos x\cos h - \sin x\sin h$

3 $$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x[\sin x]' - \sin x[\cos x]'}{\cos^2 x}$$

$$= \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \left(\frac{1}{\cos x}\right)^2$$

$$= \sec^2 x$$

4, 5, 6 $\to$ similarly using quotient rule.

Example 3. Differentiate $f(x) = \dfrac{\sec x}{1 + \tan x}$.

$$f'(x) = \frac{(1+\tan x)[\sec x]' - \sec x\,[1+\tan x]'}{(1+\tan x)^2} \quad \text{(Quotient Rule)}$$

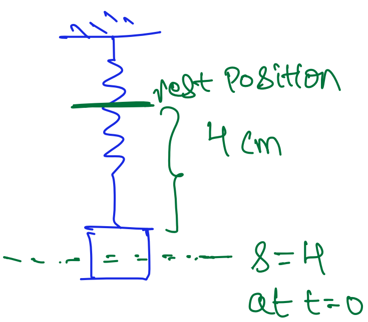
$$= \frac{(1+\tan x)\sec x \tan x - \sec x(0+\sec^2 x)}{(1+\tan x)^2}$$

$$= \frac{\sec x \tan x + \sec x \tan^2 x - \sec^3 x}{(1+\tan x)^2}$$

$\to$ can be simplified.

$$= \frac{\sec x \tan x + \sec x(\underbrace{\tan^2 x - \sec^2 x}_{=-1})}{(1+\tan x)^2} = \frac{\sec x \tan x - \sec x}{(1+\tan x)^2} = \frac{\sec x(\tan x - 1)}{(1+\tan x)^2}$$

Example 4. An object at the end of a vertical spring is stretched 4 cm beyond its rest position and released at time $t = 0$. Fixing the downward direction to be positive, its position at time t is given by $s(t) = 4\cos t$. Find the velocity and acceleration at time t. Find the time instants at which the velocity and acceleration have greatest and smallest magnitudes.



$$s(t) = 4\cos t$$

$$v(t) = s'(t) = \frac{d}{dt}(4\cos t) = 4(-\sin t)$$
$$= -4\sin t$$

$$a(t) = v'(t) = \frac{d}{dt}(-4\sin t) = -4(\cos t)$$
$$= -4\cos t$$

Observe $a(t) = -s(t)$

$\downarrow$

$s''(t) = -s(t)$

Velocity

magnitude is smallest when $\sin t = 0 \to t = n\pi$

and largest when $\sin t = \pm 1 \to t = (2n+1)\pi/2$

Acceleration

magnitude is smallest when $\cos t = 0 \to (2n+1)\pi/2$

and largest when $\cos t = \pm 1 \to n\pi$

(n is any integer)

Example 5. Find the 97-the derivative of $f(x) = \cos x$.

$$f'(x) = -\sin x$$

$$f''(x) = -\cos x$$

$$f'''(x) = -(-\sin x) = \sin x$$

$$f^{(4)}(x) = \cos x \longrightarrow f^{(8)}(x) = \cos x$$

$$f^{(4k)}(x) = \cos x$$

96 is a multiple of 4 $\Rightarrow f^{(96)}(x) = \cos x$

$$\Rightarrow f^{(97)}(x) = -\sin x$$

Example 6. Find the derivative of $r(\theta) = \theta \cos\theta$.

$$r'(\theta) = \frac{d}{d\theta}(\theta \cos\theta)$$

$$= \left[\frac{d}{d\theta}(\theta)\right] \cos\theta + \left[\frac{d}{d\theta}(\cos\theta)\right]\theta$$

$$= [1] \cos\theta + [-\sin\theta]\theta$$

$$= \cos\theta - \theta \sin\theta$$

Example 7. Find the second derivative of csc x.

$$f'(x) = -\csc x \cot x$$

$$f''(x) = \frac{d}{dx}\left[-\csc x \cot x\right]$$

$$= -\frac{d}{dx}\left[\csc x \cot x\right]$$

$$= -\frac{d}{dx}\left[\csc x\right] \cdot \cot x - \csc x \cdot \frac{d}{dx}\left[\cot x\right]$$

$$= -\left(-\csc x \cot x\right)\cot x - \csc x\left(-\csc^2 x\right)$$

$$= \csc x \cot^2 x + \csc^3 x$$

$$= \csc x\left(\cot^2 x + \csc^2 x\right)$$