Learning objectives:

1. Chain rule and its use in computing derivatives.

The Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composition $F = f \circ g$ is differentiable at x, and F' is given by

$$F'(x) = f'(g(x))\, g'(x)\ .$$

In other words, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}\ .$$

Example 1. Find $F'(x)$ if $F(x) = \sqrt{x^2 + 1}$.

$f(u) = \sqrt{u}$, $g(x) = x^2+1$

$(f \circ g)(x) = f(\underbrace{g(x)}_{u}) = f(\underbrace{x^2+1}_{u}) = \sqrt{x^2+1} = F(x).$

$F'(x) = f'(u)\, g'(x)$, $u = g(x) = x^2+1$

$= \frac{d}{du}(\sqrt{u}) \times \frac{d}{dx}[x^2+1]$

$= \frac{1}{2} u^{1/2-1} \times 2x$

$= \frac{1}{2\sqrt{u}} \times 2x = \frac{\cancel{2}x}{\cancel{2}\sqrt{u}} = \frac{x}{\sqrt{x^2+1}}$

Example 2. Differentiate

1. $y = \sin(x^2)$.
2. $y = \sin^2 x$.

① $y = \sin(x^2)$

$y = \sin u$, $u = x^2$ $\Rightarrow$ $y = \sin(x^2)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{d}{du}(\sin u) \cdot \frac{d}{dx}(x^2) = \cos u \cdot (2x)$$

$$= 2x \cos u = 2x \cos(x^2)$$

② $y = \sin^2 x$

$y = u^2$, $u = \sin x$ $\Rightarrow$ $y = (\sin x)^2 = \sin^2 x$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{d}{du}(u^2) \cdot \frac{d}{dx}(\sin x) = 2u \cdot \cos x$$

$$= 2 \sin x \cos x = \sin 2x$$

$F = f(g(x))$

The power rule combined with the chain rule

If n is any real number and $u = g(x)$, then

$$\frac{d}{dx}(u^n) = \underbrace{nu^{n-1}}_{f'(u)} \underbrace{\frac{du}{dx}}_{g'(x)}.$$

Here $f(u) = u^n$

Alternatively,

$$\frac{d}{dx}[\underset{\uparrow\, u=g(x)}{g(x)}]^n = n[g(x)]^{n-1}\, g'(x).$$

a, b are constants.

$$\frac{d}{dx}(ax+b)^n = an(ax+b)^{n-1}$$

$$\frac{d}{dx}\left[f(ax+b)\right] = a f'(ax+b)$$

$$\frac{d}{dx}(\sin(2x+1)) = 2\cos(2x+1)$$

$$\frac{d}{dx}(\tan(3x)) = 3\sec^2(3x)$$

$$\frac{d}{dx}(\cos(x+2)) = -\sin(x+2)$$

Example 3. Differentiate $y = (x^2 - 1)^{100}$.

$$\frac{dy}{dx} = 100(x^2-1)^{100-1} \cdot (x^2-1)'$$

$$= 100(x^2-1)^{99} \cdot (2x)$$

$$= 200x(x^2-1)^{99}$$

Example 4. Find $f'(x)$ if $f(x) = \dfrac{1}{\sqrt[3]{x^2 + x + 1}}$.

$$f(x) = \frac{1}{\sqrt[3]{x^2+x+1}} = \frac{1}{(x^2+x+1)^{1/3}} = (\underbrace{x^2+x+1}_{g(x)})^{-1/3 \leftarrow n}$$

$$f'(x) = \frac{-1}{3}(x^2+x+1)^{\frac{-1}{3}-1} \cdot (x^2+x+1)'$$

$$= \frac{-1}{3}(x^2+x+1)^{-4/3} \cdot (2x+1)$$

$$= \frac{-1}{3}\,\frac{1}{(x^2+x+1)^{4/3}} \cdot (2x+1)$$

$$= \frac{-(2x+1)}{3(x^2+x+1)^{4/3}}$$

Example 5. Find the derivative of the function $g(t) = \left(\frac{t-2}{2t+1}\right)^9$.

$$g(t) = u^9 \quad \text{where} \quad u = \underbrace{\frac{t-2}{2t+1}}_{\text{diff. using quotient rule.}}$$

$$g'(t) = 9\left(\frac{t-2}{2t+1}\right)^{9-1} \cdot \frac{d}{dt}\left(\frac{t-2}{2t+1}\right)$$

$$\frac{d}{dt}\left(\frac{t-2}{2t+1}\right) = \frac{(2t+1)(t-2)' - (t-2)(2t+1)'}{(2t+1)^2} = \frac{2t+1-2(t-2)}{(2t+1)^2}$$

$$= \frac{\cancel{2t}+1-\cancel{2t}+4}{(2t+1)^2} = \frac{5}{(2t+1)^2} \Rightarrow g'(t) = 9\left(\frac{t-2}{2t+1}\right)^8 \cdot \frac{5}{(2t+1)^2}$$

$$= \frac{45(t-2)^8}{(2t+1)^{10}}$$

Example 6. Differentiate $y = \underbrace{(2x+1)^5}_{f(x)}\underbrace{(x^3-x+1)^4}_{g(x)}$.

(Product rule)

$$\frac{dy}{dx} = \frac{d}{dx}\left[(2x+1)^5\right](x^3-x+1)^4 + (2x+1)^5 \frac{d}{dx}\left[(x^3-x+1)^4\right]$$

$$= 2 \times 5 \times (2x+1)^4(x^3-x+1)^4 + (2x+1)^5\left[4(x^3-x+1)^3 \cdot (x^3-x+1)'\right]$$

$$= 10(2x+1)^4(x^3-x+1)^4 + (2x+1)^5\left[4(x^3-x+1)^3(3x^2-1)\right]$$

$$= 10(2x+1)^4(x^3-x+1)^4 + 4(2x+1)^5(x^3-x+1)^3(3x^2-1)$$

$$= 2(2x+1)^4(x^3-x+1)^3\left[\underbrace{5(x^3-x+1)+2(2x+1)(3x^2-1)}_{5x^3-5x+5+2[6x^3+3x^2-2x-1]=17x^3+6x^2-9x+3}\right]$$

$$= 2(2x+1)^4(x^3-x+1)^3(17x^3+6x^2-9x+3)$$

Example 7. If $f(x) = \sin(\cos(\tan x))$, then find $f'(x)$.

Chain of length 3.

$$f(x) = \underbrace{\sin(\underbrace{\cos(\underbrace{\tan x}_{u})}_{y})}_{z}$$

$\cos(\tan x)$

$$z = \sin(y), \quad y = \cos(u), \quad u = \tan x$$

$$f'(x) = \frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \cos y \cdot (-\sin u) \cdot \sec^2 x$$

$$= -\cos y \sin u \sec^2 x$$

$$= -\cos(\cos(\tan x)) \cdot \sin(\tan x) \cdot \sec^2 x$$

Example 8. Differentiate $y = \cos\sqrt{\sin(\tan \pi x)}$.

Chain of length 5

$$y = \underbrace{\cos \underbrace{\sqrt{\underbrace{\sin(\underbrace{\tan \underbrace{\pi x}_{u}}_{v = \tan u})}_{w = \sin(v)}}}_{z = \sqrt{w}}}_{y = \cos z}$$

$$y = \cos z, \quad z = \sqrt{w}, \quad w = \sin v, \quad v = \tan u, \quad u = \pi x$$

$$z = \sqrt{\sin(\tan \pi x)}, \quad w = \sin(\tan \pi x), \quad v = \tan(\pi x)$$

$$\frac{dy}{dx} = \frac{dy}{dz} \frac{dz}{dw} \cdot \frac{dw}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx}$$

$$= -\sin z \cdot \frac{1}{2\sqrt{w}} \cdot \cos v \cdot \sec^2 u \cdot \pi$$

$$= \frac{-\pi \sin z \cos v \sec^2 u}{2\sqrt{w}}$$

$$= \frac{-\pi \sin\left(\sqrt{\sin(\tan \pi x)}\right) \cos(\tan \pi x) \sec^2(\pi x)}{2\sqrt{\sin(\tan \pi x)}}$$

Example 9. Differentiate $y = [\underbrace{x + (\underbrace{x + \sin^2 x}_{u = x+\sin^2 x})^3}_{v = x + u^3}]^4$.

$y = v^4$

$$y = v^4 = (x+u^3)^4 = (x + (x+\sin^2 x)^3)^4$$

$$\frac{dy}{dx} = \frac{d}{dx}(v^4) = 4v^3 \cdot \frac{dv}{dx}$$

$$\frac{dv}{dx} = \frac{d}{dx}(x+u^3) = 1 + \frac{d}{dx}(u^3) = 1 + 3u^2\frac{du}{dx}$$

$$\frac{du}{dx} = \frac{d}{dx}(x + \underbrace{\sin^2 x) = 1 + 2\sin x}_{\text{Chain rule}}\cos x$$

$$\frac{dy}{dx} = 4\left[x + (x+\sin^2 x)^3\right]^3\left[1 + 3(x+\sin^2 x)^2(1 + 2\sin x\cos x)\right]$$

Example 10. Differentiate $y = \sqrt{\underbrace{x + \sqrt{\underbrace{x + \sqrt{x}}_{u}}}_{v}}$.

$y = \sqrt{v}$, $v = x + \sqrt{u}$, $u = x + \sqrt{x}$

$= x + \sqrt{x+\sqrt{x}}$

$$\Rightarrow \frac{dy}{dx} = \frac{d(\sqrt{v})}{dv}\frac{dv}{dx} = \frac{1}{2\sqrt{v}}\frac{dv}{dx} = \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}\frac{dv}{dx}$$

(Chain rule)

$$\frac{dv}{dx} = \frac{d}{dx}(x) + \frac{d}{dx}(\sqrt{u}) = 1 + \frac{1}{2\sqrt{u}}\frac{du}{dx} = 1 + \frac{1}{2\sqrt{x+\sqrt{x}}}\frac{du}{dx}$$

$$\frac{du}{dx} = \frac{d}{dx}(x) + \frac{d}{dx}(\sqrt{x}) = 1 + \frac{1}{2\sqrt{x}}$$

$$\Rightarrow \frac{dv}{dx} = 1 + \frac{1}{2\sqrt{x+\sqrt{x}}}\left(1 + \frac{1}{2\sqrt{x}}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}\left[1+\frac{1}{2\sqrt{x+\sqrt{x}}}\left(1+\frac{1}{2\sqrt{x}}\right)\right]$$