Learning objectives:

1. To differentiate an equation of the form $f(x, y) = 0$ with respect to x.
2. Apply this to find equations of tangents and/or normals.

What is implicit differentiation?

When we do not have an explicit dependence of y on x like $y = f(x)$ for some function f but instead we have an equation involving both x and y. For example:

$$x^2 + y^2 + xy = 1 \ .$$

In such cases one can differentiate with respect to x to find dy/dx in terms of both x and y.

Example 1. Differentiate the following with respect to x:

1. y.
2. y^2.
3. y^3.
4. y^n.

① $\frac{d}{dx}(y) = \frac{dy}{dx}$

② $\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\,\frac{dy}{dx}$ [Chain rule]

$= 2y\,\frac{dy}{dx}$

③ $\frac{d}{dx}(y^3) = \frac{d}{dy}(y^3)\,\frac{dy}{dx}$ [Chain rule]

$= 3y^2\,\frac{dy}{dx}$

④ $\frac{d}{dx}(y^n) = \frac{d}{dy}(y^n)\,\frac{dy}{dx} = n\,y^{n-1}\,\frac{dy}{dx}$

Example 2. Differentiate the following with respect to x:

1. xy.
2. xy^2.
3. xy^3.
4. x^2y.
5. x^4y^6.

① $\frac{d}{dx}(xy) = \frac{d}{dx}(x)\, y + x\, \frac{d}{dx}(y)$ [Product rule]

$= y + x\frac{dy}{dx}$

② $\frac{d}{dx}(xy^2) = \frac{d}{dx}(x)\, y^2 + x\, \frac{d}{dx}(y^2)$ [Product rule]

$= y^2 + x\, \frac{d}{dy}(y^2)\, \frac{dy}{dx}$ [Chain rule]

$= y^2 + 2xy\frac{dy}{dx}$

③ $\frac{d}{dx}(xy^3) = \frac{d}{dx}(x)\, y^3 + x\, \frac{d}{dx}(y^3)$

chain rule

$= y^3 + 3xy^2\frac{dy}{dx}$

④ $\frac{d}{dx}(x^2y) = \frac{d}{dx}(x^2)\, y + x^2\, \frac{d}{dx}(y)$

$= 2xy + x^2\frac{dy}{dx}$

⑤ $\frac{d}{dx}(x^4y^6) = \frac{d}{dx}(x^4)\, y^6 + x^4\, \frac{d}{dx}(y^6) = 4x^3y^6 + 6x^4y^5\frac{dy}{dx}$

Example 3.

1. If $x^2 + y^2 = 25$, find dy/dx.
2. Find an equation of the tangent to the circle $x^2 + y^2 = 25$ at the point $(3, 4)$.

① $x^2 + y^2 = 25$

Diff. both sides w.r.t. x :-

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(25) \Rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

$$\Rightarrow 2x + \frac{d}{dy}(y^2)\frac{dy}{dx} = 0$$

$$\Rightarrow 2x + 2y\frac{dy}{dx} = 0$$

Solve for dy/dx :-

$$\Rightarrow 2y\frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = \frac{-2x}{2y} \Rightarrow \frac{dy}{dx} = \frac{-x}{y}$$

② eqn. of tangent at $(3,4)$

$$m_T = \left.\frac{dy}{dx}\right|_{\substack{x=3\\y=4}} = \frac{-3}{4}$$

$$\frac{y-4}{x-3} = \frac{-3}{4} \Rightarrow 4y-16 = -3x+9$$

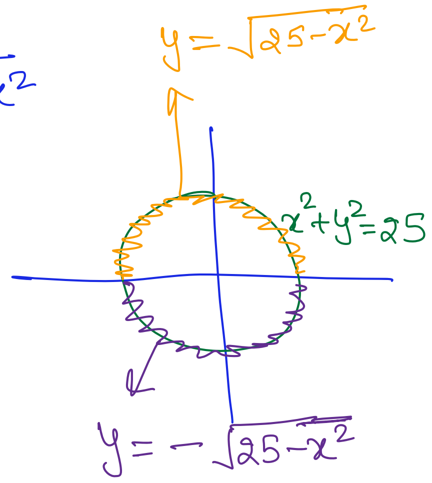
$$\Rightarrow 3x+4y-25=0$$

Alternatively

$$x^2+y^2=25 \Rightarrow y^2 = 25-x^2 \Rightarrow y = \pm\sqrt{25-x^2}$$

The Point $(3,4)$ has $y>0$

so this lies on $y=\sqrt{25-x^2}$



$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{\underbrace{25-x^2}_{u}}\right)$$

$$= \frac{d}{du}(\sqrt{u})\,\frac{du}{dx}$$

$$= \frac{1}{\cancel{2}\sqrt{25-x^2}} \times (-\cancel{2}x)$$

$$\frac{dy}{dx} = \frac{-x}{\sqrt{25-x^2}}$$

$$m_T \quad \left.\frac{dy}{dx}\right|_{x=3} = \frac{-3}{\sqrt{25-9}} = \frac{-3}{4}$$

⊛ more complicated than implicit diff.

Example 4.

1. Find y' if $x^3 + y^3 = 6xy$.
2. Find the equation tangent to the given curve at the point $(3, 3)$.
3. At what point in the first quadrant is the tangent line horizontal?
4. Find the equation of normal to the given curve at $(3, 3)$.

(1) Diff. both sides of $x^3+y^3=6xy$ w.r.t. x :-

$$\Rightarrow \frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = 6\frac{d}{dx}(xy)$$

$$\Rightarrow 3x^2 + \frac{d}{dy}(y^3)\frac{dy}{dx} = 6\left[\frac{d}{dx}(x)\,y + x\frac{dy}{dx}\right]$$

$$\Rightarrow 3x^2 + 3y^2\frac{dy}{dx} = 6\left[y + x\frac{dy}{dx}\right]$$

$$\Rightarrow \underbrace{3x^2} + \underbrace{3y^2\frac{dy}{dx}} = \underbrace{6y} + \underbrace{6x\frac{dy}{dx}}$$

Solve for dy/dx :-

$$3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} = 6y - 3x^2$$

$$\Rightarrow \frac{dy}{dx}\left(3y^2 - 6x\right) = 6y - 3x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{6y-3x^2}{3y^2-6x} = \frac{\cancel{3}(2y-x^2)}{\cancel{3}(y^2-2x)}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{2y-x^2}{y^2-2x}}$$

(2) Equation of tangent at $(3,3)$.

$$m_T = \left.\frac{dy}{dx}\right|_{\substack{x=3\\y=3}} = \frac{2(3)-(3)^2}{(3)^2-2(3)} = \frac{6-9}{9-6} = \frac{-3}{3} = -1$$

$$\frac{y-3}{x-3} = -1 \Rightarrow y-3 = -x+3$$

$$\Rightarrow x+y-6=0$$

(4) equation of normal at $(3,3)$

Perp. to tangent.

$$m_N \, m_T = -1 \Rightarrow m_N(-1) = -1$$

$$\Rightarrow m_N = 1$$

$$\frac{y-3}{x-3} = 1 \Rightarrow y-\cancel{3} = x-\cancel{3} \Rightarrow y=x$$

or $x-y=0$

(3) $$\frac{dy}{dx} = \frac{2y-x^2}{y^2-2x}$$

For horizontal tangent, $\frac{dy}{dx} = 0$

$$\frac{2y-x^2}{y^2-2x} = 0 \Rightarrow 2y-x^2 = 0$$

Solve for x and y.

Point lies on $x^3+y^3 = 6xy$

$$2y = x^2 \Rightarrow y = \frac{1}{2}x^2$$

$$x^3 + \left(\frac{1}{2}x^2\right)^3 = 6x\left(\frac{1}{2}x^2\right) \Rightarrow x^3 + \frac{1}{8}x^6 = 3x^3$$

$$\Rightarrow \frac{1}{8}x^6 - 2x^3 = 0 \Rightarrow x^3\left(\frac{1}{8}x^3 - 2\right) = 0 \Rightarrow x^3 = 0 \text{ or } \frac{1}{8}x^3 = 2$$

So, either $x^3=0$ or $x^3=16 \Rightarrow x=0$ or $x=\sqrt[3]{16}$

$y = \frac{1}{2}x^2 \Rightarrow$ either $y=0$ or $y = \frac{1}{2}\left(\sqrt[3]{16}\right)^2 = \sqrt[3]{16^2/8} = \sqrt[3]{32}$

$\Rightarrow$ required point $= (\sqrt[3]{16}, \sqrt[3]{32})$

Example 5.

1. Find y' if $\sin(x+y) = y^2\cos x - \pi^2$.
2. Find equation of tangent and normal lines to the given curve at $(0, \pi)$.

Diff. both sides of $\sin(x+y) = y^2\cos x - \pi^2$ w.r.t. x :-

$$\frac{d}{dx}\left(\sin(x+y)\right) = \frac{d}{dx}\left(y^2\cos x\right) - \frac{d}{dx}\left(\pi^2\right)$$

$\uparrow$ constant

$$\frac{d}{dx}\Big[\sin(\underbrace{x+y}_{u})\Big] = \frac{d}{dx}[\sin u] = \frac{d}{du}[\sin u]\,\frac{du}{dx}$$

$u = x+y \Rightarrow \sin(x+y) = \sin u$ (chain rule)

$$\Rightarrow \frac{d}{dx}[\sin(x+y)] = \cos u \cdot \frac{du}{dx}$$

$$= \cos(x+y)\cdot\underbrace{\frac{d}{dx}(x+y)}$$

$$= \cos(x+y)\left[1 + \frac{dy}{dx}\right] \quad \text{———} \quad (i)$$

$$\frac{d}{dx}(y^2\cos x) = y^2\,\frac{d}{dx}(\cos x) + \underbrace{\frac{d}{dx}(y^2)}_{\text{chain rule}}\cos x$$

$$= y^2(-\sin x) + \left(2y\,\frac{dy}{dx}\right)\cos x$$

$$= -y^2\sin x + 2y\cos x\,\frac{dy}{dx} \quad \text{———} \quad (ii)$$

We had

$$\underbrace{\frac{d}{dx}\left(\sin(x+y)\right)}_{(i)} = \underbrace{\frac{d}{dx}\left(y^2\cos x\right)}_{(ii)} - \underbrace{\frac{d}{dx}\left(\pi^2\right)}_{0}$$

$$\Rightarrow \cos(x+y)\left[1 + \frac{dy}{dx}\right] = -y^2\sin x + 2y\cos x\,\frac{dy}{dx}$$

Solve for $\frac{dy}{dx}$:-

$$\underbrace{\cos(x+y)} + \underbrace{\cos(x+y)\frac{dy}{dx}} = \underbrace{-y^2\sin x} + \underbrace{2y\cos x\frac{dy}{dx}}$$

$$\cos(x+y)\frac{dy}{dx} - 2y\cos x\frac{dy}{dx} = -y^2\sin x - \cos(x+y)$$

$$\frac{dy}{dx}\left[\cos(x+y) - 2y\cos x\right] = -y^2\sin x - \cos(x+y)$$

$$\frac{dy}{dx} = \frac{-y^2\sin x - \cos(x+y)}{\cos(x+y) - 2y\cos x}$$

Example 6. Find y'' if $x^4 + y^4 = 16$.

$$x^4 + y^4 = 16$$

Diff. both sides w.r.t. x :-

$$\frac{d}{dx}(x^4) + \frac{d}{dx}(y^4) = \frac{d}{dx}(16)$$

$$4x^3 + \frac{d}{dy}(y^4)\frac{dy}{dx} = 0 \Rightarrow 4x^3 + 4y^3\frac{dy}{dx} = 0$$

$$\Rightarrow 4y^3\frac{dy}{dx} = -4x^3 \Rightarrow \frac{dy}{dx} = \frac{-4x^3}{4y^3}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{-x^3}{y^3}}$$

Diff. both sides w.r.t. x :-

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{y^3(-x^3)' - (-x^3)(y^3)'}{(y^3)^2}$$

(Quotient rule)

Chain rule.

$$= \frac{y^3(-3x^2) + x^3 \overbrace{\frac{d}{dy}(y^3)\frac{dy}{dx}}}{y^6}$$

$$= \frac{-3x^2y^3 + x^3(3\cancel{y^2})\left(\frac{-x^3}{y^{\cancel{3}}}\right)}{y^6} = \frac{-3x^2y^3 - 3\frac{x^6}{y}}{y^6}$$

$$= \frac{\frac{-3x^2y^4 - 3x^6}{y}}{y^6} = \frac{-3x^2y^4 - 3x^6}{y^7} = \frac{-3x^2(\overbrace{y^4 + x^4}^{=16})}{y^7} = \frac{-48x^2}{y^7}$$

Example 7. Find y'' if $\sin y + \cos x = 1$.

Diff. both sides w.r.t. x :-

$$\frac{d}{dx}(\sin y) + \frac{d}{dx}(\cos x) = \frac{d}{dx}(1)$$

$$\frac{d}{dy}(\sin y)\frac{dy}{dx} - \sin x = 0$$

$$\Rightarrow \cos y \frac{dy}{dx} - \sin x = 0$$

$+\sin x \quad +\sin x$

$$\Rightarrow \cos y \frac{dy}{dx} = \sin x \Rightarrow \frac{dy}{dx} = \frac{\sin x}{\cos y}$$

Diff. both sides w.r.t. x :-

$$\frac{d^2y}{dx^2} = \frac{\cos y \cdot [\sin x]' - \sin x\,[\cos y]'}{(\cos y)^2}$$

$$= \frac{\cos y \cdot \cos x - \sin x \frac{d}{dy}(\cos y)\frac{dy}{dx}}{\cos^2 y}$$

$$= \frac{\cos y \cos x - \sin x(-\sin y)\frac{\sin x}{\cos y}}{\cos^2 y}$$

$$= \frac{\cos y \cos x + \frac{\sin^2 x \sin y}{\cos y}}{\cos^2 y} = \frac{\frac{\cos^2 y \cos x + \sin^2 x \sin y}{\cos y}}{\cos^2 y}$$

$$\frac{d^2y}{dx^2} = \frac{\cos^2 y \cos x + \sin^2 x \sin y}{\cos^3 y}$$