Learning objectives:

1. Applications of derivative in measuring rate of change of a quantity A relative to the rate of change in a quantity B where B is related to A through some equation.
2. Relation between A and B to be phrased in words and not as an explicit equation.

Example 1. Air is being pumped into a spherical ballon so that its volume increases at a rate of 100 cm^3/s. How fast is radius of the balloon increasing when the diameter is 50 cm.

Given : $\frac{dV}{dt} = 100$ rate of change of volume of the balloon w.r.t. time

Find : $\frac{dr}{dt}$ rate of change of radius of the balloon w.r.t time

when $r = \frac{50}{2}$ cm. $= 25$ cm.

Need a relation between V and r.

$$V = \frac{4\pi}{3} r^3 \quad \text{(volume of a sphere)}$$

Differentiate both sides w.r.t. t.

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4\pi}{3} r^3\right) = \frac{4\pi}{3} \frac{d}{dt}(r^3)$$

chain rule

$$= \frac{4\pi}{3} \frac{d}{dr}(r^3) \frac{dr}{dt}$$

$$= \frac{4\pi}{\cancel{3}} (\cancel{3} r^2) \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\Rightarrow \overset{100}{\frac{dV}{dt}} = 4\pi \overset{25}{r^2} \frac{dr}{dt} \Rightarrow 100 = 4\pi \cdot (25)^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{\cancel{100} \text{ cm}^3/\text{s}}{\cancel{4}\pi (\cancel{25})(25) \text{ cm}^2}$$

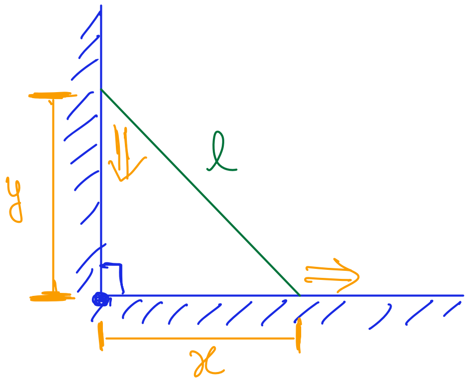
$\frac{dV}{dt}$ depends on r

$$\Rightarrow \frac{dr}{dt} = \frac{1}{25\pi} \text{ cm/s.}$$

Problem solving strategy

1. Read the problem carefully. Draw a diagram if possible.
2. Introduce notation. Assign symbols to all quantities that are functions of time.
3. Express the given information and the required rate in terms of derivatives.
4. Write and equation that related the various quantities of the problem. Use geometry, if necessary, to eliminate one of the variables by substitution.
5. Use the chain rule to differentiate both sides of the equation with respect to t.
6. Substitute given information into the resulting equation and solve for the unknown rate.

Example 2. A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?



Given $\frac{dx}{dt} = 1$ ft/s.

Rate at which x is increasing

To find $\frac{dy}{dt}$ or the rate at which y is decreasing.

when $x = 6$ft

$$6^2 + y^2 = 10^2$$

$$36 + y^2 = 100$$

$$\Rightarrow y^2 = 64 \Rightarrow y = 8 \quad (y > 0)$$

From Pythagoras theorem :-

$$x^2 + y^2 = \ell^2 = 10^2$$

$$\Rightarrow x^2 + y^2 = 100.$$

Diff. both sides w.r.t. t

$$\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dt}(100)$$

$$\frac{d}{dx}(x^2)\frac{dx}{dt} + \frac{d}{dy}(y^2)\frac{dy}{dt} = 0 \Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

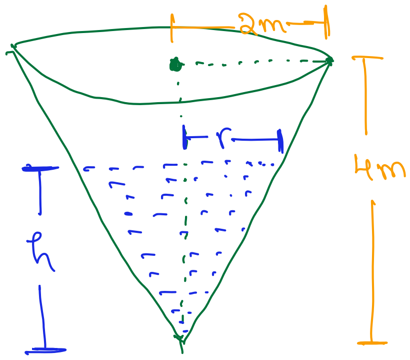
$$\Rightarrow 2y\frac{dy}{dt} = -2x\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \frac{-2x}{2y}\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

6ft, 1ft/s, 8 ft

$$\Rightarrow \frac{dy}{dt} = -\frac{6}{8}(1) = -\frac{3}{4} \text{ ft/s.}$$

y is decreasing so $dy/dt < 0$

Example 3. A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.



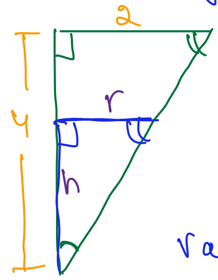
Let V be the volume of water and h be the height of water.

Given $\frac{dV}{dt} = 2 \text{ m}^3/\text{min}$.

To find $\frac{dh}{dt}$ when $h = 3$m

Volume of a right circular cone of radius r and height h is

$$V = \frac{1}{3}\pi r^2 h$$



Similar triangles $\Downarrow$ ratio of corresponding sides will be same

Need a relation between r and h.

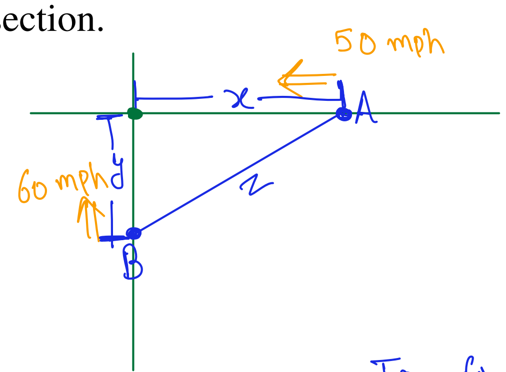
$$\frac{r}{2} = \frac{h}{4} \Rightarrow r = \frac{2h}{4} \Rightarrow r = \frac{h}{2}$$

$$\Rightarrow V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h \Rightarrow V = \frac{\pi}{12}h^3$$

Diff. w.r.t. $t \Rightarrow \underbrace{\frac{dV}{dt}}_{2\text{ m}^3/\text{min}} = \frac{\pi}{12}\frac{d}{dt}(h^3) = \frac{\pi}{12}\frac{d(h^3)}{dh}\frac{dh}{dt} = \frac{\pi}{\cancel{12}_{4}}(\cancel{3}h^2)\frac{dh}{dt} = \frac{\pi \overset{3m}{h^2}}{4}\frac{dh}{dt}$

$$\Rightarrow 2 = \frac{\pi(3)^2}{4}\,dh/dt \Rightarrow \frac{dh}{dt} = \frac{8}{9\pi} \text{ m/min.}$$

Example 4. Car A is traveling west at 50 mi/h and car B is traveling north at 60 mi/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection.



Given $\frac{dx}{dt} = -50$ mph, $\frac{dy}{dt} = -60$ mph $\Big\}$ $-$ve because x and y are both decreasing.

To find $\frac{dz}{dt}$ when $x = 0.3$ mi, $y = 0.4$ mi

By Pythagoras theorem

$$x^2 + y^2 = z^2$$

$$0.3^2 + 0.4^2 = z^2$$
$$0.09 + 0.16 = z^2 \Rightarrow z^2 = 0.25$$
$$\Rightarrow z = 0.5$$

Diff. both sides of $x^2+y^2=z^2$ w.r.t. t

$$\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dt}(z^2)$$

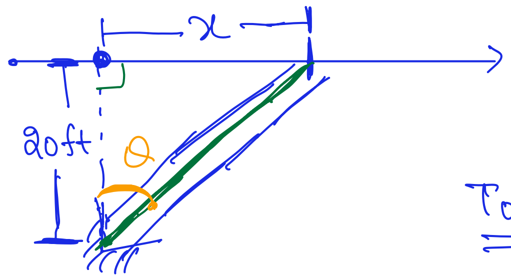
$$\frac{d}{dx}(x^2)\frac{dx}{dt} + \frac{d}{dy}(y^2)\frac{dy}{dt} = \frac{d}{dz}(z^2)\frac{dz}{dt}$$

$$\underset{0.3}{2x}\,\underset{-50}{\frac{dx}{dt}} + \underset{0.4}{2y}\,\underset{-60}{\frac{dy}{dt}} = \underset{0.5}{2z}\,\frac{dz}{dt}$$

$$\Rightarrow 2(0.3)(-50)+2(0.4)(-60) = 2(0.5)\frac{dz}{dt}$$

$$\Rightarrow \frac{dz}{dt} = -30-48 = -78 \text{ mi/h}$$

Example 5. A man walks along a straight path at a speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?



Given $\frac{dx}{dt} = 4$ ft/s

To find $\frac{d\theta}{dt}$ when $x = 15$ ft.

$$\tan\theta = \frac{x}{20} \Rightarrow x = 20\tan\theta$$

Diff. w.r.t. t :-

$$\frac{dx}{dt} = \frac{d}{dt}(20\tan\theta) = 20\frac{d}{dt}(\tan\theta)$$

Chain rule

$$= 20\frac{d}{d\theta}(\tan\theta)\frac{d\theta}{dt}$$

$$\Rightarrow \underset{4\text{ ft/s}}{\frac{dx}{dt}} = 20\sec^2\theta\,\frac{d\theta}{dt}$$

What is $\sec^2\theta$ when $x = 15$

$$\tan\theta = \frac{15}{20} = \frac{3}{4} \Rightarrow \sec^2\theta = 1 + \tan^2\theta$$

$$\Rightarrow \sec^2\theta = 1 + \left(\frac{3}{4}\right)^2 = 1 + \frac{9}{16} = \frac{25}{16}$$

$$\Rightarrow 4 = 20\left(\frac{25}{16}\right)\frac{d\theta}{dt}$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{\cancel{4} \times 16}{\underset{5}{\cancel{20}} \times 25} \text{ rad/s.}$$

$$\Rightarrow \frac{d\theta}{dt} = \boxed{\frac{16}{125} \text{ rad/s.}}$$