Learning objectives:

1. How derivative can be used to approximate nonlinear functions by linear functions.
2. Find errors and relative error in quantities.

Tangent line approximation

We can use the tangent line to approximate the curve $y = f(x)$ when x is near a.

When x is near a, we have (approximately):

$$f(x) \approx f(a) + f'(a)(x - a).$$

$$\frac{y - f(a)}{x - a} = f'(a)$$

$$\Rightarrow y - f(a) = f'(a)(x-a)$$

$$\Rightarrow y = f(a) + f'(a)(x-a)$$

The linear function

$$L(x) = f(a) + f'(a)(x - a)$$

is called the linear approximation or linearization of f at a.

Example 1. Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimates?

$$f'(x) = \frac{d}{dx}\left(\sqrt{x+3}\right) = \frac{d}{du}\left(\sqrt{u}\right)\frac{d}{dx}(x+3) = \frac{1}{2\sqrt{u}}\cdot 1 = \frac{1}{2\sqrt{x+3}}$$

$(u = x+3)$

$$f'(1) = \frac{1}{2\sqrt{1+3}} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

$$\Rightarrow L(x) = f(1) + f'(1)(x-1) = \sqrt{1+3} + \frac{1}{4}(x-1)$$

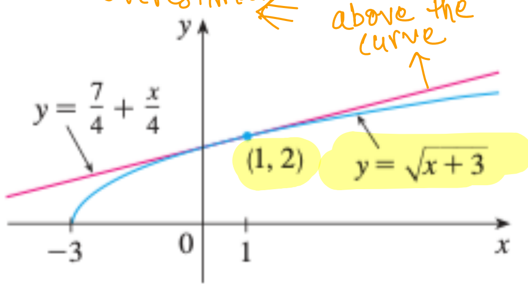
at $a=1$

$$= 2 + \frac{1}{4}x - \frac{1}{4}$$

$$\Rightarrow L(x) = \frac{7}{4} + \frac{1}{4}x$$

overestimates.

above the curve



$$\sqrt{3.98} = \sqrt{0.98+3} = f(0.98)$$

$$f(0.98) \simeq L(0.98)$$

$$= \frac{7}{4} + \frac{1}{4}(0.98) = 1.995$$

Actual value is $1.9949937\ldots$

$$\sqrt{4.05} = \sqrt{1.05+3} = f(1.05) \simeq L(1.05) = \frac{7}{4} + \frac{1}{4}(1.05) = 2.0125$$

good estimates

Actual value is $2.01246117\ldots$

Example 2. Find the linear approximations for $f(x) = \sin x$ and $g(x) = \cos x$ about the point $x = 0$.

$$L(x) = f(a) + f'(a)(x-a)$$

For $a=0$, $L(x) = f(0) + f'(0)\,x$

$\underline{\text{For } \sin x}$ $\longrightarrow f'(x) = \cos x$

$\sin(0.001) = 0.001$

$$L(x) = \sin(0) + \cos 0\,(x)$$
$$= 0 + 1\,(x)$$

$\Rightarrow L(x) = x$ or $\sin x \simeq x$ about $x=0$

$\underline{\text{For } \cos x}$ $\longrightarrow f'(x) = -\sin x$

$$L(x) = \cos 0 + (-\sin 0)\,x$$
$$= 1 + (0)x = 1$$

$\Rightarrow L(x) = 1$ or $\cos x \simeq 1$ about $x=0$

$\cos(0.001) \simeq 1$
$\cos(0.01) \simeq 1$

Differentials

Differentials are variables that take infinitesimally small values. They are denoted by putting d in front of symbols.

If $y = f(x)$ and f is a differentiable functions, then the differential dx is an independent variable and the differential dy depends on dx by the equation

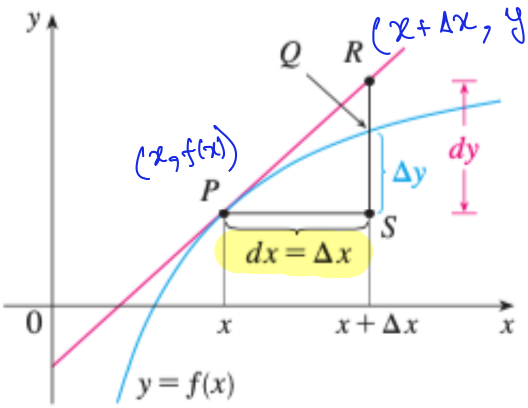
very very small changes in the variable x

very very small changes in y

$$dy = f'(x)\,dx\,.$$

differential

$(x+\Delta x,\ y+dy)$



$\shortparallel$

$(x+\Delta x,\ f(x) + f'(x)dx)$
different from Q

$Q = (x+\Delta x,\ f(x+\Delta x))$

$$\Delta y = f(x+\Delta x) - f(x)$$

$dx = \Delta x$

$$\Delta y \simeq \underbrace{dy}_{} = f'(x)\,dx$$

used to approximate $\underbrace{\text{(small) changes}}_{\text{errors}}$ in y

Example 3. Compare the values Δy and dy if $y = f(x) = x^3 + x^2 - 2x + 1$ and x changes

1. from 2 to 2.05, and
2. from 2 to 2.01.

$$f'(x) = 3x^2 + 2x - 2 \Rightarrow f'(2) = 3(2)^2 + 2(2) - 2 = 14$$

① $dy = f'(x)\,dx, \quad dx = \Delta x = 2.05 - 2 = 0.05$

$$dy = f'(\underset{\text{initial value of } x}{2})(0.05) = 14(0.05) = 0.7$$

$$f(2) = (2)^3 + (2)^2 - 2(2) + 1 = 9, \quad f(2.05) = (2.05)^3 + (2.05)^2 - 2(2.05) + 1 = 9.717625$$

$$\Delta y = f(2.05) - f(2) = 9.717625 - 9 = 0.717625$$

close estimate

0.7

② $dx = \Delta x = 2.01 - 2 = 0.01$

$$dy = f'(2)(0.01) = 0.14$$

0.14

close estimate

$$\Delta y = f(2.01) - f(2) = 9.140701 - 9 = 0.140701$$

Example 4. The radius of a sphere is measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm. What is the maximum error in volume if we use this value of radius to compute it.

$$r = 21 \text{ cm}, \quad \underbrace{dr}_{\text{error}} \leq 0.05 \text{ cm}$$

$V = \frac{4\pi}{3} r^3$. To find: dV when $r = 21$ cm, $dr = 0.05$ cm

$$\Rightarrow dV = V'(r) \cdot dr, \quad V'(r) = \frac{d}{dr}\left(\frac{4\pi}{3} r^3\right) = \frac{4\pi}{3} \frac{d}{dr}(r^3) = 4\pi r^2$$

$$\Rightarrow dV = 4\pi r^2\, dr$$

$$= 4\pi (21)^2 (0.05) \text{ cm}^3$$

$$\simeq 277 \text{ cm}^3 \Rightarrow \text{Maximum error in volume is } 277 \text{ cm}^3$$

$$\frac{dV}{V} = \frac{\cancel{4\pi} r^2\, dr}{\frac{\cancel{4\pi}}{3} r^3} = \frac{3\,dr}{r} = \frac{3(0.05)}{21}$$

$$= \frac{0.15}{21} \simeq \underline{\underline{0.07}}$$

Relative error

Relative error in a quantity y is given by $\dfrac{dy}{y}$.

Percentage (relative) error in a quantity y is given by $\dfrac{dy}{y} \times 100\ \%$.

Example 5. The relative error in the radius of a sphere is 0.0024%. Find the relative error in the volume of the sphere if the same (erroneous) value of radius is used to compute the volume.

this determines the ratio of relative errors.

$$V = \frac{4\pi}{3} r^3 \Rightarrow dV = 4\pi r^2\, dr$$

$$\frac{dV}{V} = \frac{4\pi r^2\, dr}{\frac{4\pi}{3} r^3} = 3\,\frac{dr}{r}$$

$$= 3 \times 0.0024\ \%$$

$$= 0.072\ \%$$

Example 6. The area of a circle was measured and it was found that the measured value has a relative error of 1%. If we compute radius of the circle using this value of area, what would be the relative error in the radius of the circle.

$$A = \pi r^2$$

$$\Rightarrow dA = \frac{d}{dr}(\pi r^2) \cdot dr$$

$$= 2\pi r \, dr$$

$$\Rightarrow \underbrace{\frac{dA}{A}}_{1\%} = \frac{2\cancel{\pi}\cancel{r}\, dr}{\cancel{\pi} r^{\cancel{2}}} = \frac{2\, dr}{r} = 2\underbrace{\left(\frac{dr}{r}\right)}_{\text{to find}}$$

$$\Rightarrow \frac{dr}{r} = \frac{1}{2}\frac{dA}{A}$$

$$= \frac{1}{2}(1\%) = 0.5\%$$