Learning objectives:

1. Learn the concept of **absolute** maximum and minimum points/values of a function.
2. Learn the concept of **local** maximum and minimum points/values of a function.
3. The Extreme value theorem and the Fermat's theorem.
4. Critical numbers of a function.
5. The closed interval method.

Absolute maximum and minimum

Let c be a number in the domain D of a function f. Then $f(c)$ is the

1. absolute maximum value of f on D if $f(c) \geq f(x)$ for all x in D.
2. absolute minimum value of f on D if $f(c) \leq f(x)$ for all x in D.

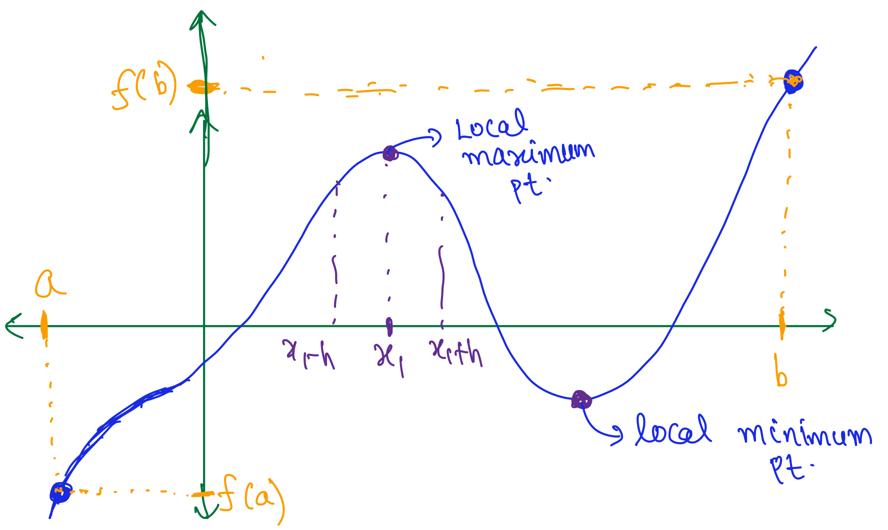
Local maximum and minimum

Let c be a number in the domain D of a function f. Then $f(c)$ is the

1. local maximum value of f on D if $f(c) \geq f(x)$ when x is near c.
2. local minimum value of f on D if $f(c) \leq f(x)$ when x is near c.

There exists some number $h>0$ so that on $[c-h, c+h]$, $f(c)$ is maximum.

There exists some number $h>0$ so that on $[c-h, c+h]$, $f(c)$ is minimum

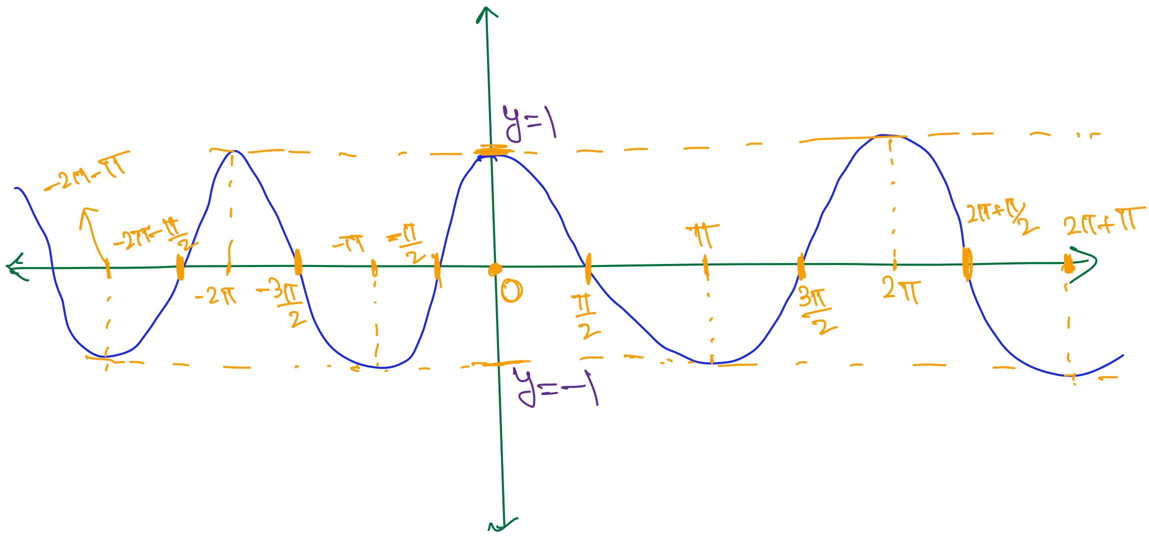


On $[a,b]$, $f(a)$ is absolute minimum value.

On $[a,b]$, $f(b)$ is absolute maximum value.

Example 1. Find absolute maximum and minimum values.

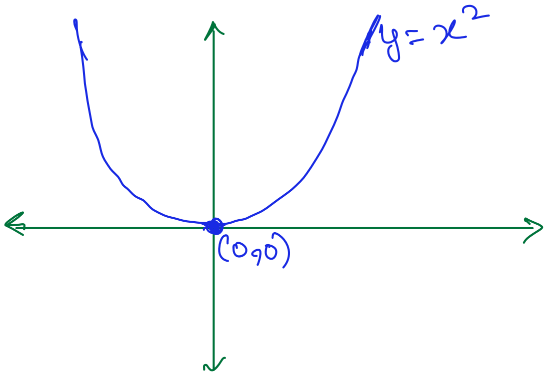
1. $y = \cos x$.
2. $y = x^2$.
3. $y = x^3$.



Absolute max value = +1

Absolute min value = −1

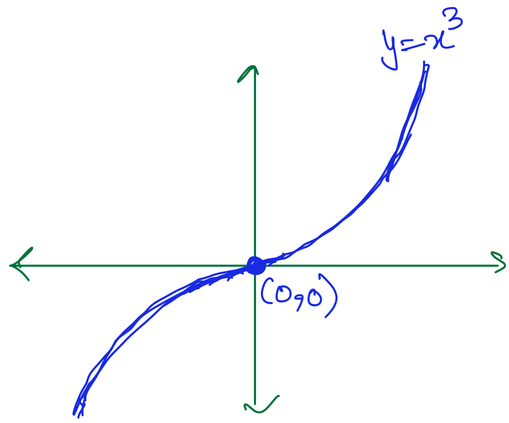
②



Absolute min value = 0

Absolute max. value = ∞

③

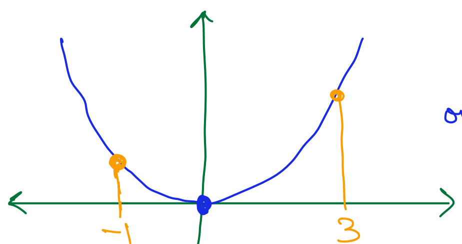


Absolute min value = −∞

Absolute max value = ∞

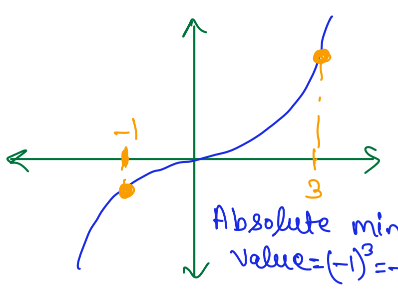
The Extreme value theorem.

If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.



Absolute min value = 0

$y=x^2$ on $[-1,3]$ has $3^2=9$ as an absolute maximum value.



Absolute min Value $=(-1)^3=-1$

Absolute max value $= (3)^3 = 27$

Fermat's Theorem

If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

• Let c be a local maximum pt.

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

when $c+h > c$ then $h > 0$

since c is local max. point $\Rightarrow f(c+h) \leq f(c)$

$$\frac{\overbrace{f(c+h) - f(c)}^{\to -ve}}{\underbrace{h}_{\to +ve}} \text{ is } -ve \Rightarrow \underset{x=c}{RHL} \leq 0$$

when $c+h < c$ then $h < 0$ but $f(c+h) \leq f(c)$

$$\frac{\overbrace{f(c+h) - f(c)}^{-ve}}{\underbrace{h}_{\to -ve}} \text{ is } +ve \Rightarrow \underset{x=c}{LHL} \geq 0$$

Since $f'(c)$ exist, $\underset{x=c}{RHL} = \underset{x=c}{LHL} = 0$

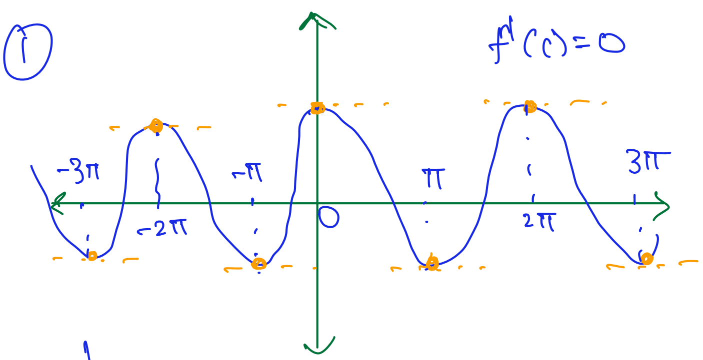
$\Rightarrow f'(c) = 0$

Example 2.

1. $y = \cos x$.
2. $y = x^2$.
3. $y = x^3$.
4. $y = |x|$.

Find local max/min Pts.

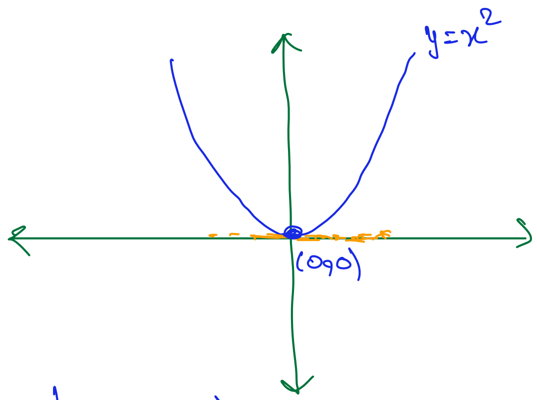
①



$$y' = \frac{d}{dx}(\cos x) = -\sin x = 0$$

$\sin x = 0 \Rightarrow x = n\pi$ and also local max/min pts.

②

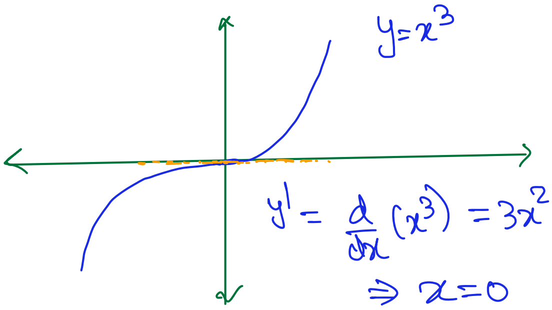


$$y' = \frac{d}{dx}(x^2) = 2x = 0$$

$\Rightarrow x = 0$ only solution.

and also a local min pt.

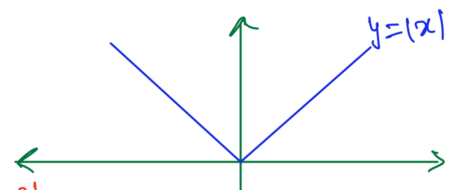
③



$$y' = \frac{d}{dx}(x^3) = 3x^2 = 0$$

$\Rightarrow x = 0$

④



$f'(0)$ does not exist

But we still have $x=0$

$y'=0$ at $x=0$ but $x=0$ is not a local max/min pt. to be a local min pt.

All the Potential Pts. of local maximum/minimum

Critical number

A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Example 3. Find the critical numbers of the following functions.

1. $f(x) = x^{3/5}(4 - x)$.
2. $f(x) = 2x^3 - 3x^2 - 36x$.
3. $g(t) = |3t - 4|$.

1 $f'(x) = \frac{d}{dx}\left[x^{3/5}(4-x)\right] = \left[x^{3/5}\right]'(4-x) + x^{3/5}[4-x]'$

(Product rule) $= \frac{3}{5}x^{3/5-1}(4-x) + x^{3/5}(-1)$

$= \frac{3}{5}x^{-2/5}(4-x) - x^{3/5} = \frac{3(4-x)}{5x^{2/5}} - x^{3/5}$

$= \frac{3(4-x) - 5x^{2/5}(x^{3/5})}{5x^{2/5}} = \frac{12-3x-5x}{5x^{2/5}}$

$\Rightarrow f'(x) = \frac{12-8x}{5x^{2/5}} \Rightarrow f'(x)=0 \Rightarrow \frac{12-8x}{5x^{2/5}} = 0 \Rightarrow 12-8x=0$

$\Rightarrow 12 = 8x \Rightarrow x = \frac{12}{8} \Rightarrow \boxed{x = \frac{3}{2}}$ $f'\left(\frac{3}{2}\right)=0$

$x^{2/5} = 0 \Rightarrow \boxed{x=0}$ $f'(0)$ does not exist.

2 $f'(x) = 2(3x^2) - 3(2x) - 36 = 6x^2 - 6x - 36 = 6(x^2-x-6)$

$f'(x) = 0 \Rightarrow 6(x^2-x-6) = 0 \Rightarrow x^2-x-6 = 0 \Rightarrow (x-3)(x+2) = 0$

$x^2-3x+2x-6 \to x(x-3)+2(x-3)$

$\Rightarrow x-3=0$ or $x+2=0$

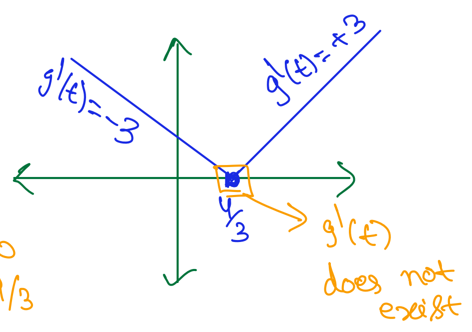
$\Rightarrow \boxed{x=3}$ or $\boxed{x=-2}$ (two critical numbers)

3 $g(t) = |3t-4| = \begin{cases} 3t-4 & \text{if } t \geq 4/3 \\ -3t+4 & \text{if } t < 4/3 \end{cases}$

$3t-4=0 \Rightarrow t = 4/3$

Since $g'(4/3)$ does not exist,

$\boxed{t = \frac{4}{3}}$ is a critical number.



Fermat's theorem rephrased

If f has a local maximum or minimum at c, then c is a critical number of f.

The closed interval method

To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the critical numbers of f in the open interval (a, b). $\rightarrow x_1, x_2, \ldots, x_k$
2. Find the values of f at the critical numbers of f in (a, b). $\rightarrow f(x_1), f(x_2), \ldots, f(x_k)$
3. Find the values of f at the endpoints, that is, find $f(a)$ and $f(b)$.
4. The largest of the values from steps 2 and 3 is the absolute maximum value; the smallest of these values is the absolute minimum value.
   $\hookrightarrow$ Compare $f(a), f(x_1), \ldots, f(x_k), f(b)$

Example 4. Find the absolute maximum and minimum values of the given function on the given interval.

$$f(x) = x^3 - 3x^2 + 1\,, \quad -\frac{1}{2} \le x \le 4\,.$$

(1) Find critical numbers

$$f'(x) = 3x^2 - 6x = 3x(x-2)$$

$$f'(x) = 0 \Rightarrow 3x(x-2) = 0 \Rightarrow x = 0 \text{ or } x - 2 = 0$$

$$\Rightarrow x = 0 \text{ or } x = 2$$

(2) Find values of f on critical numbers.

$$f(0) = 0^3 - 3(0)^2 + 1 = 1$$

$$f(2) = 2^3 - 3(2)^2 + 1 = 8 - 12 + 1 = -3$$

(3) Find f(a) and f(b).

$$a = \frac{-1}{2},\ b = 4 \Rightarrow f\left(\frac{-1}{2}\right) = \left(\frac{-1}{2}\right)^3 - 3\left(\frac{-1}{2}\right)^2 + 1 = \frac{-1}{8} - \frac{3}{4} + 1 = \frac{1}{8}$$

$$f(4) = 4^3 - 3(4)^2 + 1 = 64 - 48 + 1 = 17$$

(4) $\left\{f(0), f(2), f\left(\frac{-1}{2}\right), f(4)\right\} = \left\{1, -3, \frac{1}{8}, 17\right\} \Rightarrow$ Absolute max value $= 17$

Absolute min value $= -3$

Example 5. Find the absolute maximum and minimum values of the given function on the given interval.

$$f(x) = x - 2\sin x\,, \quad 0 \le x \le 2\pi\,.$$

① Critical numbers

$f'(x) = 1 - 2\cos x$

$f'(x) = 0 \Rightarrow 1 - 2\cos x = 0 \Rightarrow 2\cos x = 1 \Rightarrow \cos x = \frac{1}{2}$

For $0 \le x \le 2\pi$, $\cos x > 0$ in 1^{st} and 4^{th} quadrant.

$\Rightarrow$ In 1^{st} quadrant, $\cos x = \frac{1}{2} \Rightarrow x = \frac{\pi}{3}$

$\Rightarrow$ In 4^{th} quadrant, $\cos x = \frac{1}{2} \Rightarrow x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

② Critical values

$f(\pi/3) = \frac{\pi}{3} - 2\sin\frac{\pi}{3} = \frac{\pi}{3} - 2\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3} - \sqrt{3} \approx 1.04 - 1.73 = -0.69$

$f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} - 2\underbrace{\sin\frac{5\pi}{3}}_{\sin(2\pi - \pi/3)} = \frac{5\pi}{3} - 2\left(-\sin\pi/3\right) = \frac{5\pi}{3} + 2\,\frac{\sqrt{3}}{2}$

$= \frac{5\pi}{3} + \sqrt{3} \approx 5.20 + 1.73 = 6.93$

③ f(a) and f(b)

$f(0) = 0 - 2\sin 0 = 0$

$f(2\pi) = 2\pi - 2\sin(2\pi) = 2\pi \approx 6.28$

④ Absolute max value $= \frac{5\pi}{3} + \sqrt{3} \approx 6.93$

Absolute min value $= \pi/3 - \sqrt{3} \approx -0.69$

Example 6. Find the absolute maximum and minimum values of the given function on the given interval.

$$f(x) = x + \frac{1}{x}\,, \quad [-1.5, -0.5] \cup [0.5, 1.5]\,.$$

① Critical numbers

$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2} \Rightarrow f'(x) = 0 \Rightarrow \frac{x^2-1}{x^2} = 0 \Rightarrow x^2 - 1 = 0$

$\Rightarrow x^2 = 1$

$\Rightarrow x = \pm 1$

Denominator of $f'(x) = x^2 = 0$

$\Rightarrow \boxed{x = 0}$ ← not included in $[-1.5, -0.5] \cup [0.5, 1.5]$ so ignore

② Critical values

$f(1) = 1 + \frac{1}{1} = 2$ and $f(-1) = -1 - \frac{1}{1} = -2$

③ Values at end points $\Rightarrow f(-1.5) = -1.5 - \frac{1}{1.5} = -1.5 - 0.67 = -2.17$

$f(-0.5) = -0.5 - \frac{1}{0.5} = -2.5$

$f(1.5) = 2.17$, $f(0.5) = 2.5$

$f(-x) = -x - \frac{1}{x} = -\left(x + \frac{1}{x}\right) = -f(x)$

④ Compare

Absolute max value $= 2.5$

Absolute min value $= -2.5$

Example 7. Find the absolute maximum and minimum values of the given function on the given interval.

$$f(\theta) = 2\cos\theta + \sin 2\theta\,, \quad [0, \pi/2]\,.$$

① Critical numbers

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$

$f'(\theta) = -2\sin\theta + 2\cos 2\theta$

$= -2\sin\theta + 2(1 - 2\sin^2\theta) = 2 - 2\sin\theta - 4\sin^2\theta$

$= -2\underbrace{(2\sin^2\theta + \sin\theta - 1)}_{= 2x^2 + x - 1}$. Let $\sin\theta = x$

$f'(\theta) = 0 \Rightarrow 2\sin^2\theta + \sin\theta - 1 = 0$ or $2x^2 + x - 1 = 0$

$\Rightarrow 2x^2 + 2x - x - 1 = 0$

$\Rightarrow 2x(x+1) - 1(x+1) = 0$

$\Rightarrow (2x-1)(x+1) = 0 \Rightarrow x = \frac{1}{2}, -1$

$\Rightarrow \sin\theta = \frac{1}{2}$

or

$\boxed{\sin\theta = -1}$ not possible for $0 \le \theta \le \pi/2$

$\Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

② Critical values: $f(\pi/6) = 2\cos\frac{\pi}{6} + \sin\left(2\cdot\frac{\pi}{6}\right)$

$= 2\cos\frac{\pi}{6} + \sin\frac{\pi}{3} = 2\left(\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$

$= \frac{3 \times 1.732}{2} \approx 3(0.866) = 2.598$

③ Values at end points

$f(0) = 2\cos 0 + \sin 2(0) = 2(1) + 0 = 2$

$f\left(\frac{\pi}{2}\right) = 2\cos\frac{\pi}{2} + \sin\left(2\cdot\frac{\pi}{2}\right) = 2\cos\frac{\pi}{2} + \sin\pi = 2(0) + 0 = 0$

(4) Compare

Absolute max value $= \frac{3\sqrt{3}}{2}$

Absolute min value $= 0$