Learning objectives:

1. Rolle's theorem.
2. The Mean value theorem.
3. Applications.

Rolle's Theorem

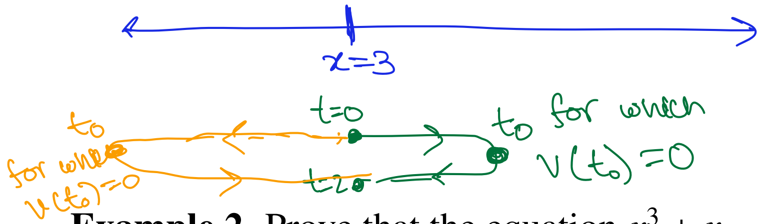
Let f be a function that satisfies the following three conditions:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$.

Then there is a number c in (a, b) such that $f'(c) = 0$.

continuously and smoothly.

Example 1. An object is moving in a straight line along the x-axis. Suppose the object was at position $x = 3$ at $t = 0$ and at $t = 2$. Show that at some instant between $t = 0$ and $t = 2$, the object was at rest.



$v(t) = s'(t)$, s is continuous and differentiable on $[0,2]$

$s(0) = s(2) = 3$

$\Rightarrow$ There is some $0 < c < 2$ such that $s'(c) = 0$

requride to

Example 2. Prove that the equation $x^3 + x - 1$ has exactly one real root.

Intermediate value theorem

$f(0) = 0^3 + 0 - 1 = -1$

$f(1) = 1^3 + 1 - 1 = 1$

$f(x) = x^3 + x - 1$

$f'(x) = 3x^2 + 1 \geq 1$

And f is continuous. 0 is between $f(0) = -1$ and $f(1) = 1$

So, by intermediate value theorem, there must be some number c b/w 0 and 1 $(0 < c < 1)$ so that $f(c) = 0$

$\Rightarrow x^3 + x - 1 = 0$ has at least one solution. or f has at least one real root.

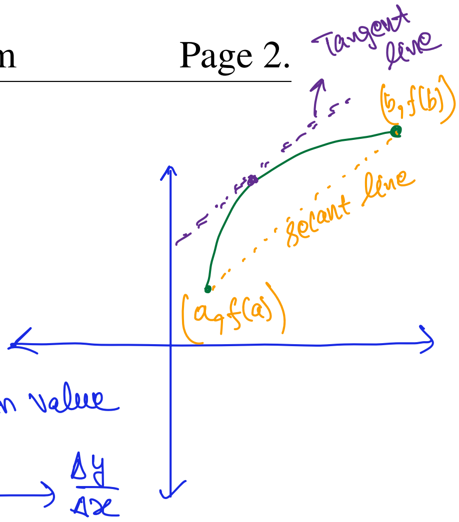
Suppose there are two roots: c and d. Assume $c \neq d$.

$\Rightarrow f(c) = f(d) = 0$

By Rolle's theorem we must have some number a between c and d such that $f'(a) = 0 \to$ Not possible because $f'(x) \geq 1$.

$\Rightarrow$ By contradiction f has exactly one real root.

The Mean Value Theorem



Let f be a function that satisfies the following two conditions:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

instantaneous velocity   average or mean value velocity.

$$\left.\frac{dy}{dx}\right|_{x=c} \leftarrow \underbrace{f'(c)}_{\text{slope of tangent}} = \underbrace{\frac{f(b) - f(a)}{b - a}}_{} , \longrightarrow \frac{\Delta y}{\Delta x}$$

slope of line joining endpoints.

or equivalently

$y=f(x)$

$$\underbrace{f(b) - f(a)}_{\text{net change in } y} = f'(c)\underbrace{(b - a)}_{\text{net change in } x} .$$

Example 3. Let $f(x) = x^3 - x$, $a = 0$, $b = 2$. Check/illustrate that the mean value theorem holds.

① $f(x)$ is continuous on $[0,2]$
② $f(x)$ is differentiable on $(0,2)$ } f is a polynomial

$$\boxed{\frac{2\times 1.732}{3} = 2(0.577) = 1.154}$$

$$\frac{f(b)-f(a)}{b-a} = \frac{f(2)-f(0)}{2-0} = \frac{(2^3-2)-0}{2} = 3$$

$$f'(c) = 3c^2 - 1 \Rightarrow 3c^2 - 1 = 3 \Rightarrow 3c^2 = 4 \Rightarrow c^2 = \frac{4}{3} \Rightarrow c = \pm\frac{2}{\sqrt{3}}$$

$\pm\frac{2\sqrt{3}}{3}$ ॥

$0 < c < 2$ ⇓ Theorem holds

Example 4. Suppose $f(0) = -3$ and $f'(x) \le 5$ for all values of x. How large can $f(2)$ possibly be?

By the Mean value Theorem (MVT), we have

$$\frac{f(2)-f(0)}{2-0} = f'(c) \le 5$$

$$\Rightarrow \frac{f(2)-(-3)}{2} \le 5$$

$$\Rightarrow f(2)+3 \le 10$$

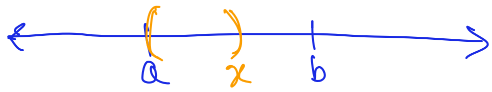
$$\Rightarrow f(2) \le 7$$

$\Rightarrow$ The largest $f(2)$ can be $\rightarrow$ is 7

Example 5. If $f'(x) = 0$ for all x in an interval (a, b), then show that f is constant on (a, b).

Take the closed interval $[a, x]$ where $a < x < b$

$$\frac{f(x) - f(a)}{x - a} = \underset{\substack{\parallel \\ 0}}{f'(c)} \quad \text{for some } c \text{ between } a \text{ and } x$$



$$\Rightarrow \frac{f(x) - f(a)}{x - a} = 0 \Rightarrow f(x) - f(a) = 0 \quad (x - a \neq 0)$$

$$\Rightarrow f(x) = f(a) \text{ for every } a < x < b$$

$\Rightarrow$ f is a constant on (a, b)

Example 6. If $f'(x) = g'(x)$ for all x in an interval (a, b), then show that $f(x) = g(x) + c$ for some constant c whenever $a < x < b$.

Let $h(x) = f(x) - g(x)$

$$\Rightarrow h'(x) = f'(x) - g'(x)$$
$$= 0 \quad \text{for } a < x < b$$

From Example 5, we have $h(x) = c$ for some constant c when $a < x < b$

$$\Rightarrow f(x) - g(x) = c \Rightarrow f(x) = g(x) + c$$

Example 7. Does there exist a function f such that $f(0) = -1$, $f(2) = 4$ and $f'(x) \le 2$ for all x?

f is continuous and differentiable

Apply MVT on $[0,2]$

$$\frac{f(2)-f(0)}{2-0} = f'(c) \quad \text{for some} \quad 0<c<2$$

$$\frac{4-(-1)}{2} = f'(c) \Rightarrow f'(c) = \frac{5}{2} \Rightarrow f'(c) = 2.5 \text{ for at least one } c.$$

But we have $f'(x) \le 2 \Rightarrow f'(x)$ cannot be 2.5

This is a contradiction.

$\Rightarrow$ Such an f cannot exist.

Example 8. Two runners start a race at the same time and finish in a tie. Prove that at some time during the race they have the same speed.

Velocity $\rightarrow v_1(t)$, $v_2(t)$. Let displacement functions be $s_1(t), s_2(t)$

Start at time $t=a$ and end at time $t=b$

Let $s(t) = s_1(t) - s_2(t)$

$$\left.\begin{array}{ll} \text{At } t=a, & s_1(a) = s_2(a) \\ \text{At } t=b, & s_1(b) = s_2(b) \end{array}\right] \Rightarrow s(a)=0,\ s(b)=0$$

$$\frac{s(b)-s(a)}{b-a} = \frac{0-0}{b-a} = 0 = s'(c)$$

By MVT, there must be some instant c between a and b

such that $s'(c) = 0 \Rightarrow s_1'(c) - s_2'(c) = 0 \Rightarrow v_1(c) - v_2(c) = 0$

$\Rightarrow v_1(c) = v_2(c)$