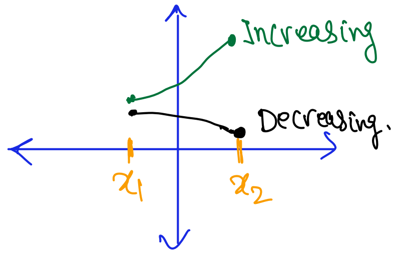
Learning objectives:

1. Increasing and decreasing functions, and their relation to derivative.
2. The first derivative test for local extremal values.
3. What is concavity and convexity?
4. What are inflection points?
5. The second derivative test.

graph of a function.

What is meant by being increasing/decreasing?



Let $x_1 < x_2$.

$f(x_1) \leq f(x_2)$ $\Rightarrow$ f is increasing

$f(x_1) \geq f(x_2)$ $\Rightarrow$ f is decreasing

Increasing/Decreasing Test

1. If $f'(x) > 0$ on an interval, then f is increasing on that interval.
2. If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Example 1. Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

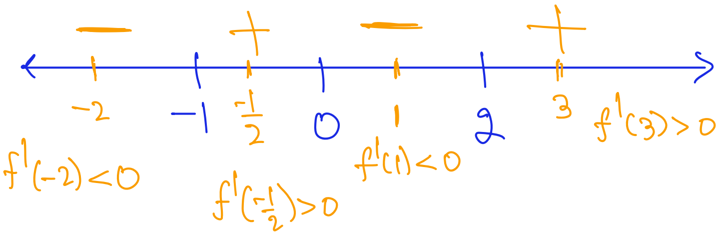
↳ on what intervals.

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2)$$

$$x^2 - x - 2 = x^2 - 2x + x - 2 = x(x-2) + 1(x-2)$$
$$= (x+1)(x-2)$$

$$f'(x) = 12x(x+1)(x-2)$$

$$f'(x) = 0 \Rightarrow x = 0,\ x = -1,\ x = 2$$



finding the sign of $f'(x)$ on each interval

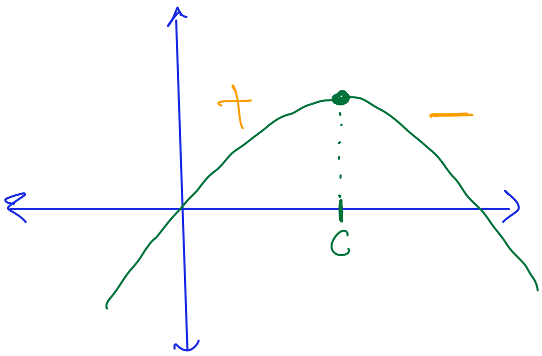
f is increasing on $(-1, 0) \cup (2, \infty)$

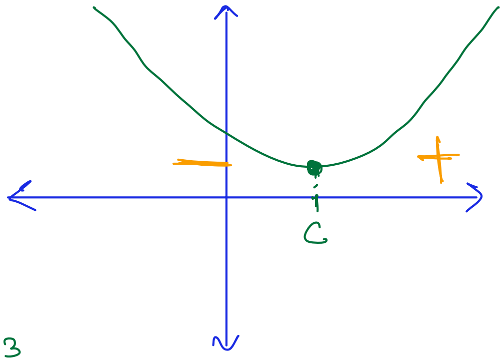
f is decreasing on $(-\infty, -1) \cup (0, 2)$

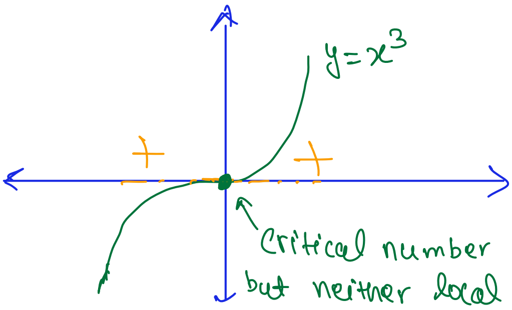
The First Derivative Test

Suppose that c is a critical number of a continuous function f.

1. If f' changes sign from positive to negative at c, then f has a local max at c.
2. If f' changes sign from negative to positive at c, then f has a local min at c.
3. If f' does not change sign at c, then f has neither local max nor min at c.







$$\frac{dy}{dx} = 3x^2$$

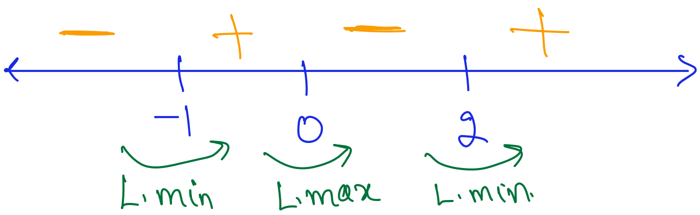
$$\left.\frac{dy}{dx}\right|_{x=0} = 0$$

Critical number but neither local max nor local min.

Example 2. Find the local minimum and maximum values of the function f in Example 1.

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

$$f'(x) = 12x(x+1)(x-2)$$



finding the sign of $f'(x)$ on each interval

L. max. values : $f(0) = 5$

L. min values : $f(-1) = 3(-1)^4 - 4(-1)^3 - 12(-1)^2 + 5$

$= 3 + 4 - 12 + 5 = 0$

$f(2) = 3(2)^4 - 4(2)^3 - 12(2)^2 + 5$

$= 48 - 32 - 48 + 5 = -27$

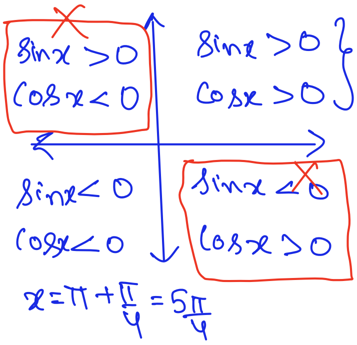
Example 3. Find the local maximum and minimum values of $f(x) = \sin x + \cos x$, $0 \le x \le 2\pi$.

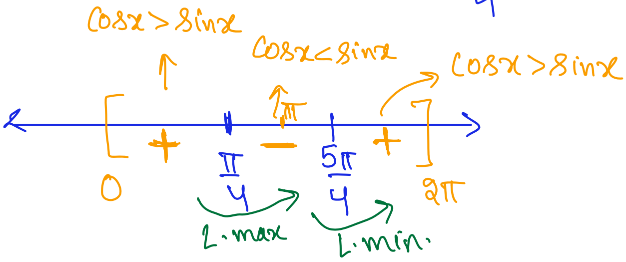
$$f'(x) = \cos x - \sin x$$

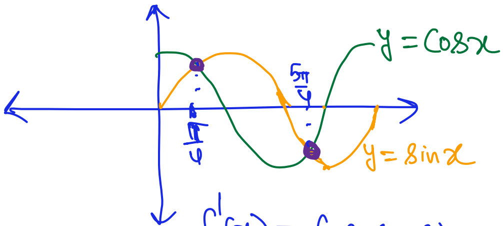
$$f'(x) = 0 \Rightarrow \cos x - \sin x = 0 \Rightarrow \sin x = \cos x$$

$$\Rightarrow \tan x = 1$$

$$\Rightarrow x = \frac{\pi}{4}, \quad x = \frac{5\pi}{4}$$







$f'(x) = \cos x - \sin x$

Alternatively, to determine sign choose some points in each interval.

$\cos 0 - \sin 0 = 1 - 0 = 1 > 0$

$\cos\pi - \sin\pi = -1 - 0 = -1 < 0$

$\cos 2\pi - \sin 2\pi = 1 - 0 = 1 > 0$

L. max value $f(\pi/4) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$

L. min. value $f(5\pi/4) = \sin 5\pi/4 + \cos 5\pi/4 = -1/\sqrt{2} - \frac{1}{\sqrt{2}} = -\sqrt{2}$

Example 4. Find the local maximum and minimum values of $f(x) = \dfrac{x^2}{x-1}$.

$$f'(x) = \frac{(x-1)[x^2]' - x^2[(x-1)]'}{(x-1)^2} \quad \text{[Quotient Rule]}$$

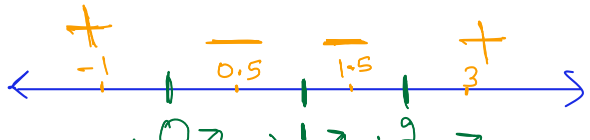
$$= \frac{(x-1)(2x) - x^2(1)}{(x-1)^2} = \frac{2x^2 - 2x - x^2}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2}$$

$$\Rightarrow f'(x) = \frac{x(x-2)}{(x-1)^2}$$

Critical numbers: $f'(x) = 0 \Rightarrow x(x-2) = 0$

$\Rightarrow x = 0$ or $x = 2$

$f'(x)$ does not exist $\Rightarrow (x-1)^2 = 0 \Rightarrow x = 1$



$f'(-1) > 0$ $f'(3) > 0$

$f'(1.5) < 0$

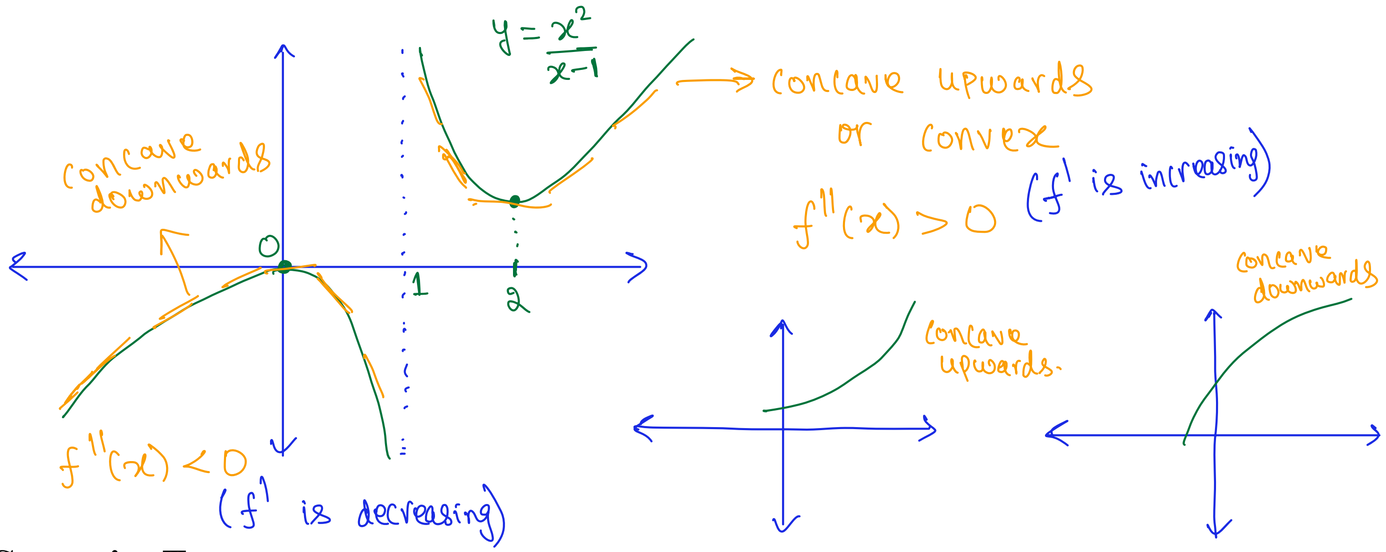
$f'(0.5) < 0$

Local max values: $f(0) = \frac{0^2}{0-1} = 0$ ← max

Local min values: $f(2) = \frac{2^2}{2-1} = 4$ ← min.

Concavity and Convexity

If the graph of f lies above all of its tangent lines on an interval I, then it is called concave upward or convex on I. If the graph of f lies below all of its tangent lines on I, it is called concave downward on I.



Concavity Test

1. If $f''(x) > 0$ on an interval I, then f is concave upward on I.
2. If $f''(x) < 0$ on an interval I, then f is concave downward on I.

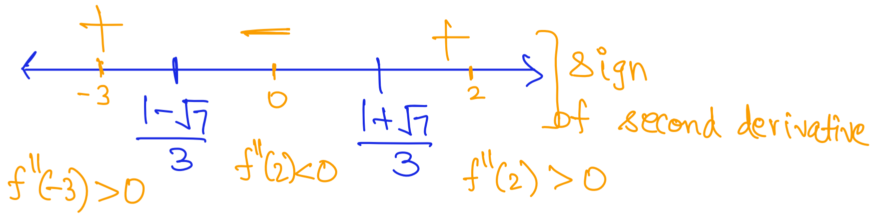
Example 5. Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is concave upward and where it is concave downward.

$$f'(x) = 12x^3 - 12x^2 - 24x$$

$$f''(x) = 36x^2 - 24x - 24 = 12(3x^2 - 2x - 2)$$

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(3)(-2)}}{2(3)} = \frac{2 \pm \sqrt{4+24}}{6} = \frac{2 \pm \sqrt{28}}{6}$$

$$\Rightarrow x = \frac{2 \pm 2\sqrt{7}}{6} = \frac{1 \pm \sqrt{7}}{3}$$

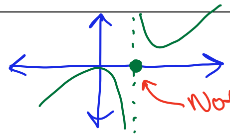


f is concave upwards in $\left(-\infty, \frac{1-\sqrt{7}}{3}\right) \cup \left(\frac{1+\sqrt{7}}{3}, \infty\right)$

f is concave downwards in $\left(\frac{1-\sqrt{7}}{3}, \frac{1+\sqrt{7}}{3}\right)$

$$\boxed{\circledast \ x = \frac{1 \pm \sqrt{7}}{3} \text{ are the two inflection pts. of } f.}$$

Inflection Point



Not inflection Pt. because f not continuous there.

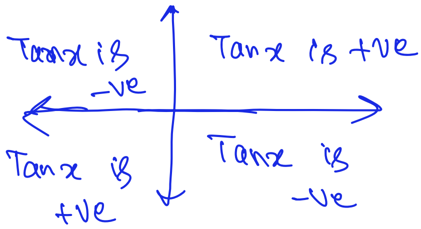
A point P on a curve $y = f(x)$ is called an inflection point if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

Example 6. Find the inflection points of $f(x) = \sin x + \cos x$, $0 \le x \le 2\pi$.

$$f'(x) = \cos x - \sin x \Rightarrow f''(x) = -\sin x - \cos x$$

$$\Rightarrow f''(x) = 0 \Rightarrow -\sin x - \cos x = 0 \Rightarrow -\sin x = \cos x$$

$$\Rightarrow -\tan x = 1 \Rightarrow \underbrace{\tan x = -1}_{\text{Pts. in } 0 \le x \le 2\pi}$$

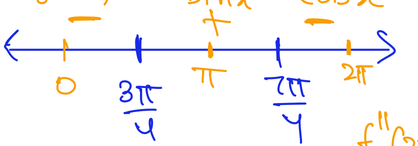


$\tan x = 1$ when $x = \frac{\pi}{4}$

$\Rightarrow \tan x = -1$ when $x = \pi - \frac{\pi}{4}$ or $x = 2\pi - \frac{\pi}{4}$

$\Rightarrow \left\{ x = \frac{3\pi}{4} \text{ or } x = \frac{7\pi}{4} \right\}$ $f''(x) = 0$

$f''(x) = -\sin x - \cos x$



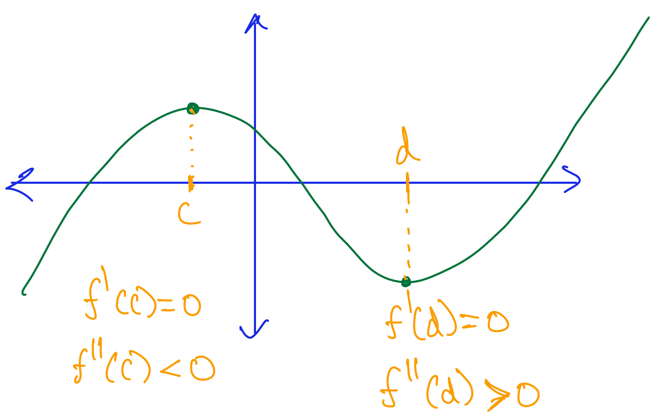
$f''(0) = -1$ $\quad f''(\pi) = 1 > 0$ $\quad f''(2\pi) = -1 < 0$

$\Rightarrow$ Inflection Pts. are $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$

The second derivative test

Suppose f'' is continuous near c.

1. If $f'(c) = 0$ and $f''(c) > 0$ then f has a local minimum at c.
2. If $f'(c) = 0$ and $f''(c) < 0$ then f has a local maximum at c.



Example 7. Let $y = x^4 - 4x^3$. Find the intervals of concavity, points of inflections, and local maximum and minimum points of the given curve. Use this information to sketch the curve.

$f'(x) = 4x^3 - 12x^2 \Rightarrow$ Critical numbers

$f'(x) = 0 \Rightarrow 4x^3 - 12x^2 = 0 \Rightarrow 4x^2(x-3) = 0$

$f''(x) = 12x^2 - 24x$

$\Rightarrow x = 0$ or $x = 3$

Second derivative test

$f''(0) = 12(0)^2 - 24(0) = 0 \Rightarrow x=0$ is neither L.min nor L.max.

$f''(3) = 12(3)^2 - 24(3) = 108 - 72 = 36 > 0$

$f''(3) > 0 \Rightarrow x = 3$ is a local min.

(*) No maximum pt

(*) One minimum pt.
$x = 3$

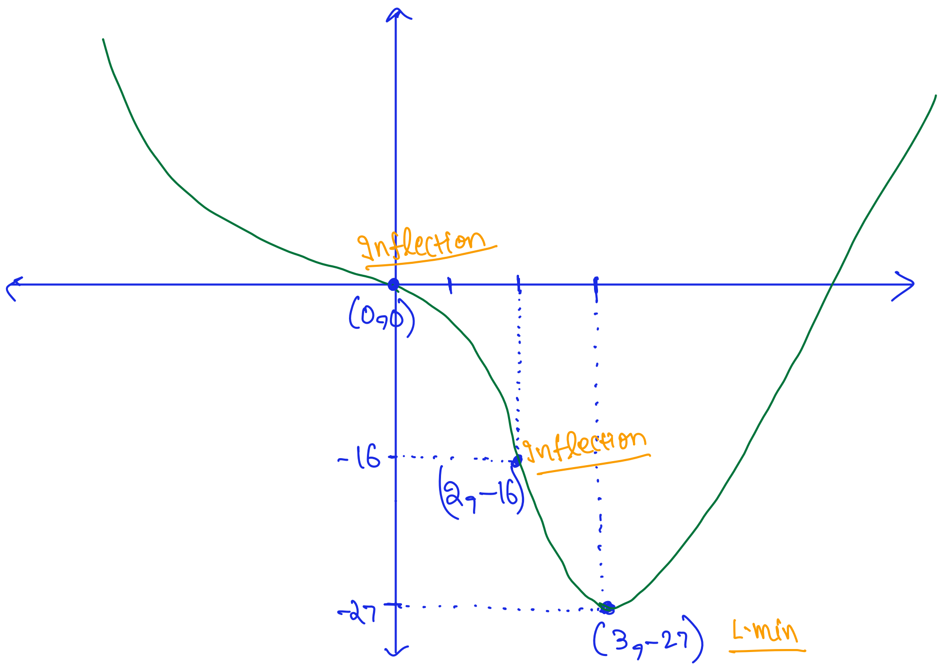
$f(3) = 3^4 - 4(3)^3 = 81 - 108 = -27$

$f''(x) = 12x(x-2) = 0 \Rightarrow x = 0, 2$

+ − +
0 2
Concave up, Concave down, Concave up

(*) Pts. of inflection are :-
$x = 0$ and $x = 2$

$f(0) = 0^4 - 4(0)^3 = 0$, $f(2) = 2^4 - 4(2)^3 = 16 - 32 = -16$



Example 8. Sketch the graph of the function $f(x) = x^{2/3}(6-x)^{1/3}$.

$$f'(x) = [x^{2/3}]'(6-x)^{1/3} + x^{2/3}\left[(6-x)^{1/3}\right]'$$

$$= \frac{2}{3}x^{\frac{2}{3}-1}(6-x)^{1/3} + x^{2/3}\left[\frac{1}{3}(6-x)^{\frac{1}{3}-1}\right]\frac{d}{dx}(6-x)$$

$$= \frac{2}{3}x^{-1/3}(6-x)^{1/3} + \frac{1}{3}x^{2/3}(6-x)^{-2/3}(-1)$$

$$= \frac{2(6-x)^{1/3}}{3x^{1/3}} - \frac{x^{2/3}}{3(6-x)^{2/3}} = \frac{2(6-x)-x}{3x^{1/3}(6-x)^{2/3}}$$

$$= \frac{12-2x-x}{3x^{1/3}(6-x)^{2/3}} = \frac{12-3x}{3x^{1/3}(6-x)^{2/3}} = \frac{-3(x-4)}{3x^{1/3}(6-x)^{2/3}}$$

$$= \frac{-(x-4)}{x^{1/3}(6-x)^{2/3}}$$

$\Rightarrow$ critical numbers $x=4$] $f'(x)=0$; $x=0$, $x=6$] $f'(x)$ d.n.e.

$$f''(x) = \frac{x^{1/3}(6-x)^{2/3}[-(x-4)]' + (x-4)\left[x^{1/3}(6-x)^{2/3}\right]'}{x^{2/3}(6-x)^{4/3}}$$

$$\left[x^{1/3}(6-x)^{2/3}\right]' = \frac{1}{3}x^{-2/3}(6-x)^{2/3} - \frac{2}{3}x^{1/3}(6-x)^{-1/3}$$

$$= \frac{(6-x)^{2/3}}{3x^{2/3}} - \frac{2x^{1/3}}{3(6-x)^{1/3}} = \frac{6-x-2x}{3x^{2/3}(6-x)^{1/3}} = \frac{6-3x}{3x^{2/3}(6-x)^{1/3}}$$

$$= \frac{-3(x-2)}{3x^{2/3}(6-x)^{1/3}} = \frac{-(x-2)}{x^{2/3}(6-x)^{1/3}}$$

$$f''(x) = \frac{-x^{1/3}(6-x)^{2/3} - \dfrac{(x-4)(x-2)}{x^{2/3}(6-x)^{1/3}}}{x^{2/3}(6-x)^{4/3}} = \frac{-x(6-x)-(x-4)(x-2)}{x^{4/3}(6-x)^{5/3}}$$

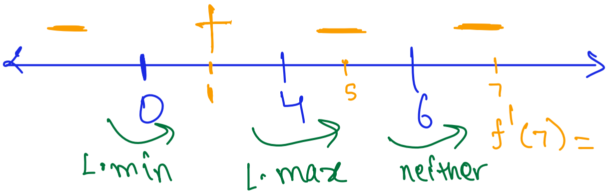
$$= \frac{-6x+x^2-[x^2-6x+8]}{x^{4/3}(6-x)^{5/3}} = \frac{-8}{x^{4/3}(6-x)^{5/3}}$$

$$f'(x) = \frac{-(x-4)}{x^{1/3}(6-x)^{2/3}}$$

$\Rightarrow$ Critical numbers

$x=4$] $f'(x)=0$

$x=0$, $x=6$] $f'(x)$ d.n.e.



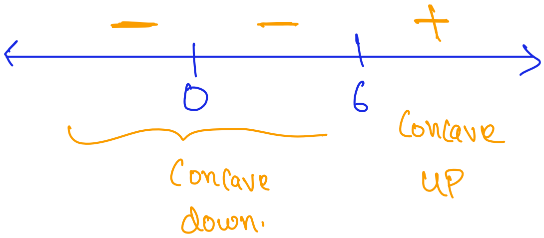
$$f'(7) = \frac{-(7-4)}{7^{1/3}(-1)^{2/3}} = \frac{-3}{7^{1/3}} < 0$$

$$f'(5) = \frac{-(5-4)}{5^{1/3}(1)^{2/3}} = \frac{-1}{5^{1/3}} < 0$$

$$f'(1) = \frac{-(1-4)}{(6-1)^{2/3}} = \frac{-(-3)}{5^{2/3}} > 0$$

$$f'(-1) = \frac{-(-1-4)}{(-1)^{1/3}(7)^{2/3}} = \frac{5}{(-1)7^{2/3}} < 0$$

$$f''(x) = \frac{-8}{x^{4/3}(6-x)^{5/3}}$$

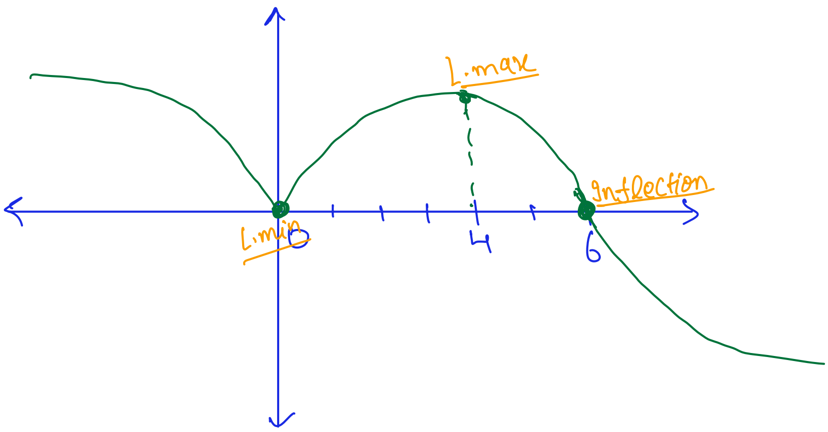


$$f(x) = x^{2/3}(6-x)^{1/3}$$

$f(0) = 0$

$f(4) = 4^{2/3}\, 2^{1/3} > 0$

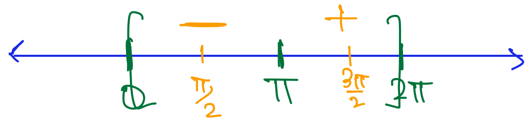
$f(6) = 0$



Example 9. Sketch the graph of the function $f(\theta) = 2\cos\theta + \cos^2\theta, \quad 0 \le \theta \le 2\pi.$

$$f'(\theta) = -2\sin\theta + 2\cos\theta(-\sin\theta) = -2\sin\theta(1+\cos\theta)$$

$$f'(\theta) = 0 \Rightarrow \sin\theta = 0 \text{ or } \cos\theta = -1 \Rightarrow \theta = 0, \pi, 2\pi$$



$f'(\pi/2) = -2 < 0 \quad f'(\frac{3\pi}{2}) = 2 > 0$

$\Rightarrow x = \pi$ is pt. of l.min.

$f(\pi) = 2(-1) + (-1)^2 = -1$

$$f''(\theta) = -2\cos\theta \underbrace{- 2\cos^2\theta + 2\sin^2\theta}_{}$$

obtained from product rule on $-2\cos\theta\sin\theta$

$$= -2\cos\theta - 2\cos^2\theta + \underbrace{2 - 2\cos^2\theta}_{}$$

$\sin^2\theta = 1 - \cos^2\theta$

$$= 2 - 2\cos\theta - 4\cos^2\theta = -2(2\cos^2\theta + \cos\theta - 1)$$

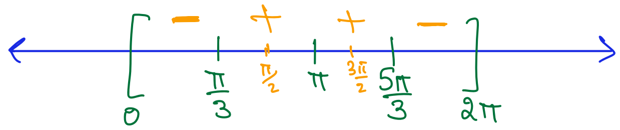
$$= -2[2\cos^2\theta + 2\cos\theta - \cos\theta - 1] = -2[2\cos\theta(\cos\theta+1) - 1(\cos\theta+1)]$$

$$= -2(2\cos\theta - 1)(\cos\theta + 1)$$

$$f''(\theta) = 0 \Rightarrow \underbrace{\cos\theta = \frac{1}{2}}_{} \text{ or } \underbrace{\cos\theta = -1}_{}$$

$$\Rightarrow \theta = \frac{\pi}{3} \text{ or } 2\pi - \frac{\pi}{3} \quad \text{or} \quad \theta = \pi$$

$$= \pi/3 \text{ or } \frac{5\pi}{3} \quad \text{or } \pi$$



$\Rightarrow x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$ are pts. of inflection.

$f''(0) < 0, \ f''(\pi/2) > 0, \ f''(\frac{3\pi}{2}) > 0, \ f''(2\pi) < 0$

$$f(\pi/3) = f(\tfrac{5\pi}{3}) = 2(\tfrac{1}{2}) + (\tfrac{1}{2})^2$$

$$= 1 + \frac{1}{4} = \frac{5}{4}$$

$f(0) = 3$

$f(2\pi) = 3$