Learning objectives:

1. Understand the concept of limits at infinity.
2. Find horizontal asymptotes to a curve.

Intuitive definition of a limit at infinity.

Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = L$$

means the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.

Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x\to-\infty} f(x) = L$$

means the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.

Horizontal asymptote.

The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if

$$\text{either} \quad \lim_{x\to-\infty} f(x) = L \quad \text{or} \quad \lim_{x\to\infty} f(x) = L\,.$$

Example

Example 1. Find $\lim_{x\to\infty} \frac{1}{x}$ and $\lim_{x\to-\infty} \frac{1}{x}$

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

Take any small number, let's say $c = 0.0001$

we can produce an x such that $\frac{1}{x} = 0.0001$

Pick a sufficiently large value for x

Just choose x to be larger than 10^4

$x > 10^4 \Rightarrow \frac{1}{x} < 10^{-4} = c$

$\lim_{x\to-\infty} \frac{1}{x} = 0$ because for any small number $c = -0.0001$

$x < -10^4 \Rightarrow \frac{1}{x} > -0.0001$

Theorem

If $r > 0$ is a rational number, then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0\,.$$

$\lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$

$\lim_{x\to-\infty} \frac{1}{\sqrt[3]{x}} = 0$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x\to-\infty} \frac{1}{x^r} = 0\,.$$

The limit laws are valid for limits at infinity as well (with the exception of direct substitution).

Example 2. Evaluate

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}.$$

Divide both the numerator and the denominator by the highest power of x in denominator

(in this case x^2)

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2+4x+1} = \lim_{x\to\infty} \frac{\frac{3x^2}{x^2} - \frac{x}{x^2} - \frac{2}{x^2}}{\frac{5x^2}{x^2} + \frac{4x}{x^2} + \frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}} = \frac{\lim_{x\to\infty} 3 - \lim_{x\to\infty}\frac{1}{x} - 2\lim_{x\to\infty}\frac{1}{x^2}}{\lim_{x\to\infty} 5 + 4\lim_{x\to\infty}\frac{1}{x} + \lim_{x\to\infty}\frac{1}{x^2}}$$

$$= \frac{3 - 0 - 2(0)}{5 + 4(0) + 0} = \frac{3}{5}$$

Example 3. Find the horizontal and vertical asymptotes to the graph of the function

$$f(x) = \frac{\sqrt{2x^2+1}}{3x-5}.$$

Vertical Asymptotes

$3x - 5 = 0 \Rightarrow 3x = 5 \Rightarrow x = \frac{5}{3}$ is a vertical asymptote.

Horizontal Asymptotes

Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5}$$ (Divide both numerator and denominator by x)

$$= \lim_{x\to\infty} \frac{\frac{1}{x}\sqrt{2x^2+1}}{\frac{1}{x}(3x-5)} = \lim_{x\to\infty} \frac{\sqrt{\frac{1}{x^2}}\sqrt{2x^2+1}}{\frac{3x}{x} - \frac{5}{x}} = \lim_{x\to\infty} \frac{\sqrt{\frac{1}{x^2}(2x^2+1)}}{3 - \frac{5}{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{2x^2}{x^2}+\frac{1}{x^2}}}{3-\frac{5}{x}} = \lim_{x\to\infty} \frac{\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}} = \frac{\sqrt{\lim_{x\to\infty} 2 + \lim_{x\to\infty}\frac{1}{x^2}}}{\lim_{x\to\infty} 3 - 5\lim_{x\to\infty}\frac{1}{x}}$$

$$= \frac{\sqrt{2+0}}{3-5(0)} = \frac{\sqrt{2}}{3} \Rightarrow y = \frac{\sqrt{2}}{3} \text{ is a horizontal asymptote}$$

$$\lim_{x\to-\infty} \frac{\frac{1}{x}\sqrt{2x^2+1}}{\frac{1}{x}(3x-5)} = \lim_{x\to-\infty} \frac{-\sqrt{\frac{1}{x^2}}\sqrt{2x^2+1}}{\frac{3x}{x}-\frac{5}{x}} = \lim_{x\to-\infty} \frac{-\sqrt{\frac{2x^2}{x^2}+\frac{1}{x^2}}}{3-\frac{5}{x}}$$

$$= -\frac{\sqrt{\lim_{x\to-\infty} 2 + \lim_{x\to-\infty}\frac{1}{x^2}}}{\lim_{x\to-\infty} 3 - 5\lim_{x\to-\infty}\frac{1}{x}} = -\frac{\sqrt{2+0}}{3-5(0)} = -\frac{\sqrt{2}}{3} \Rightarrow y = -\frac{\sqrt{2}}{3} \text{ is a horizontal asymptote}$$

Example 4. Compute $\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)$.

$\infty + 1 = \infty$

$$= \lim_{x\to\infty}\sqrt{x^2+1} - \lim_{x\to\infty} x = \underbrace{\infty - \infty}_{} \neq 0$$

(Indeterminate form)

$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)\frac{\left(\sqrt{x^2+1}+x\right)}{\left(\sqrt{x^2+1}+x\right)} = \lim_{x\to\infty}\frac{\left(\sqrt{x^2+1}-x\right)\left(\sqrt{x^2+1}+x\right)}{\sqrt{x^2+1}+x}$$

$$= \lim_{x\to\infty}\frac{\left(\sqrt{x^2+1}\right)^2-(x)^2}{\sqrt{x^2+1}+x} = \lim_{x\to\infty}\frac{x^2+1-x^2}{\sqrt{x^2+1}+x}$$

$$= \lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x}$$

Divide both numerator and denominator by x

$$= \lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{x}\sqrt{x^2+1}+\frac{x}{x}} = \lim_{x\to\infty}\frac{\frac{1}{x}}{\sqrt{\frac{x^2+1}{x^2}}+1}$$

$$= \lim_{x\to\infty}\frac{\frac{1}{x}}{\sqrt{1+\frac{1}{x^2}}+1} = \frac{\lim_{x\to\infty}\frac{1}{x}}{\sqrt{\lim_{x\to\infty}1+\lim_{x\to\infty}\frac{1}{x^2}}+\lim_{x\to\infty}1} = \frac{0}{\sqrt{1+0}+1}$$

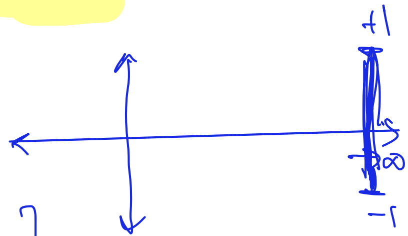
$$= \frac{0}{2} = 0$$

Example 5. Evaluate $\lim_{x\to\infty} \sin\frac{1}{x}$.

Let $t = \frac{1}{x}$. When $x \to \infty$, $t \to 0$ $\qquad \lim_{x\to\infty}\frac{1}{x} = 0$

$$\lim_{x\to\infty} \sin\frac{1}{x} = \lim_{t\to 0} \sin t = \sin(0) = 0$$

Example 6. Evaluate $\lim_{x\to\infty} \sin x$.



$\sin(x+2\pi) = \sin x$ [Periodic function]

$\infty + 2\pi = \infty$, $\infty + a = \infty$ for any $0 \le a \le 2\pi$

$\lim_{x\to\infty} \sin x \sim$ oscillates between -1 and 1

$\lim_{x\to\infty} \cos x \sim [-1, 1]$

$\lim_{x\to 0} \sin\frac{1}{x} \sim [-1, 1]$

Infinite Limits at Infinity

We write

$$\lim_{x\to\infty} f(x) = \infty$$

when values of $f(x)$ become arbitrarily large as x becomes large.

Similarly, we can define

$$\lim_{x\to-\infty} f(x) = \infty\,, \quad \lim_{x\to\infty} f(x) = -\infty\,, \quad \lim_{x\to-\infty} f(x) = -\infty\,.$$

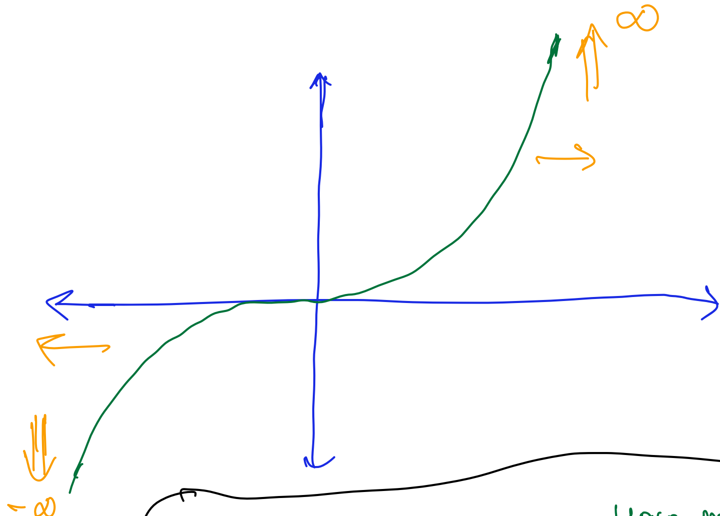
Example 7. Find $\lim_{x\to\infty} x^3$ and $\lim_{x\to-\infty} x^3$.

Given any positive number M

we can choose

$x > \sqrt[3]{M}$, so that

$f(x) = x^3 > M$



$\Rightarrow \lim_{x\to\infty} x^3 = \infty$

$\Rightarrow \lim_{x\to-\infty} x^3 = -\infty$

$\lim_{x\to\infty} x^{\frac{m}{n}} = \infty$ Here $m>0$, $n>0$

$\lim_{x\to-\infty} x^{\frac{m}{n}} = -\infty$ m is odd.

In the case, m is even, $\lim_{x\to-\infty} x^{m/n} = \infty$

Example 8. Find $\lim_{x\to\infty}(x^2 - x)$.

$$= \lim_{x\to\infty} x^2 - \lim_{x\to\infty} x = \boxed{\infty - \infty}$$ indeterminate form.

$$= \lim_{x\to\infty}(x^2 - x) = \lim_{x\to\infty} x(x-1)$$

$$= \lim_{x\to\infty} x \lim_{x\to\infty}(x-1) = \infty\cdot\infty = \infty$$

Note $\lim_{x\to\infty}(x-x^2) = \lim_{x\to\infty} x(1-x)$

$$= \lim_{x\to\infty} x \lim_{x\to\infty}(1-x) = -\infty$$

Example 9. Find $\lim_{x\to\infty}\dfrac{x^2+x}{3-x}$.

Divide both num and den by highest power in denominator.

$$\lim_{x\to\infty}\frac{x^2+x}{3-x} = \lim_{x\to\infty}\frac{\frac{1}{x}(x^2+x)}{\frac{1}{x}(3-x)} = \lim_{x\to\infty}\frac{x+1}{\frac{3}{x}-1}$$

$$= \frac{\lim_{x\to\infty} x + \lim_{x\to\infty} 1}{\lim_{x\to\infty}\frac{3}{x} - \lim_{x\to\infty} 1} = \frac{\infty+1}{0-1} = \frac{\infty}{-1} = -\infty$$

Example 10. Find $\lim_{x\to\infty}\dfrac{x}{x^2+1}$.

Divide both num and den by highest power of x in denominator

$$\lim_{x\to\infty}\frac{x}{x^2+1} = \lim_{x\to\infty}\frac{\frac{1}{x^2}(x)}{\frac{1}{x^2}(x^2+1)} = \lim_{x\to\infty}\frac{\frac{1}{x}}{1+\frac{1}{x^2}}$$

$$= \frac{\lim_{x\to\infty}\frac{1}{x}}{\lim_{x\to\infty} 1 + \lim_{x\to\infty}\frac{1}{x^2}} = \frac{0}{1+0} = \frac{0}{1} = 0$$

In general

$$\lim_{x\to\infty}\frac{P(x)}{q(x)} = \begin{cases} 0 & \text{if } \deg P < \deg q \\ \text{finite nonzero} & \text{if } \deg P = \deg q \\ \pm\infty & \text{if } \deg P > \deg q. \end{cases}$$

Horizontal Asymptote