**Learning objectives**:

1. To draw a rough sketch of the graph of a given function, highlighting important points.

## Guidelines for sketching a curve

1. Domain

   Range of $x$-values where we draw the graph.

2. Intercepts → intersection with $x$ and $y$ axes.

   y-intercept : ⇒ $x = 0$ ⇒ $f(0)$ gives the y-intercept

   x-intercepts : ⇒ Solve $f(x) = 0$ for $x$.

3. Symmetry

   about y-axis.
   if $f$ is even function
   that is, $f(-x) = f(x)$

   about the origin
   if $f$ is odd function
   that is, $f(-x) = -f(x)$.

4. Asymptotes

   Find all horizontal and all vertical asymptotes.

5. Intervals of increase or decrease

   $\underbrace{f'(x) > 0}_{\text{increasing}}$ or $\underbrace{f'(x) < 0}_{\text{decreasing}}$.

6. Local maximum and minimum values

   → Critical numbers and use (5).

7. Concavity and points of inflection

   → $\underbrace{f''(x) > 0}_{\text{concave up}}$ or $f''(x) < 0$ concave down

8. Combining the above information to sketch the curve

   ↓
   highlight important points.

**Example 1**. Sketch the curve $y = \dfrac{2x^2}{x^2 - 1}$.

① Domain : $x^2 - 1 \neq 0 \Rightarrow x \neq \pm 1$

② Intercepts : y-intercept : $f(0) = 0 \Rightarrow (0, 0)$

x-intercept : $f(x) = 0 \Rightarrow \dfrac{2x^2}{x^2-1} = 0 \Rightarrow 2x^2 = 0 \Rightarrow x = 0$

③ Symmetry $f(-x) = \dfrac{2(-x)^2}{(-x)^2 - 1} = \dfrac{2x^2}{x^2-1} = f(x) \Rightarrow$ even function (symm. about y-axis)

④ Asymptotes

Vertical $\longrightarrow$ Put denominator equal to 0 $\Rightarrow x^2 - 1 = 0 \Rightarrow x = 1$, $x = -1$

Horizontal $\longrightarrow \displaystyle\lim_{x \to \infty} \frac{2x^2}{x^2-1} = \lim_{x \to -\infty} \frac{2x^2}{x^2-1}$

$\Rightarrow \displaystyle\lim_{x \to \infty} \frac{2x^2}{x^2-1} = \lim_{x \to \infty} \frac{2}{1 - \frac{1}{x^2}} = \frac{2}{1-0} = 2 \Rightarrow y = 2$

⑤ Increasing/Decreasing :

$$f'(x) = \frac{(x^2-1)[2x^2]' - [(x^2-1)]'(2x^2)}{(x^2-1)^2} = \frac{4x(x^2-1) - (2x)(2x^2)}{(x^2-1)^2}$$

$$= \frac{4x^3 - 4x - 4x^3}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$$

$+$ | $-$

Increasing | 0 | Decreasing.

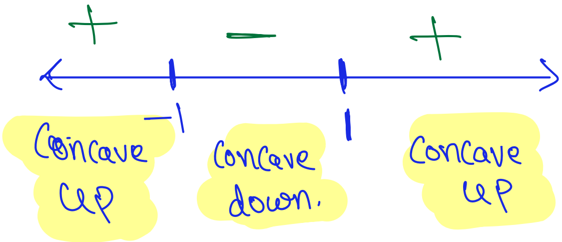
⑥ L. max/min : $x = 0$ is pt. of L. max, $f(0) = 0$

⑦ Concavity : $f''(x) = \dfrac{(x^2-1)^2[-4x]' - (-4x)[(x^2-1)^2]'}{(x^2-1)^4}$

$$= \frac{-4(x^2-1)^2 + 4x[2(x^2-1)(2x)]}{(x^2-1)^4} = \frac{-4(x^2-1)^2 + 16x^2(x^2-1)}{(x^2-1)^4}$$

$$= \frac{4\cancel{(x^2-1)}[-(x^2-1) + 4x^2]}{(x^2-1)^{\cancel{4}\,3}} = \frac{4[-x^2+1+4x^2]}{(x^2-1)^3}$$

$$\Rightarrow f''(x) = \frac{4(3x^2+1)}{(x^2-1)^3} = \frac{\overbrace{4(3x^2+1)}^{\text{always +ve}}}{\underbrace{(x^2-1)^2}_{\text{always +ve}}\,\underbrace{(x^2-1)}_{\text{can change sign}}}$$




sign of $f''(x)$ is same as the sign of $x^2-1 = (x-1)(x+1)$

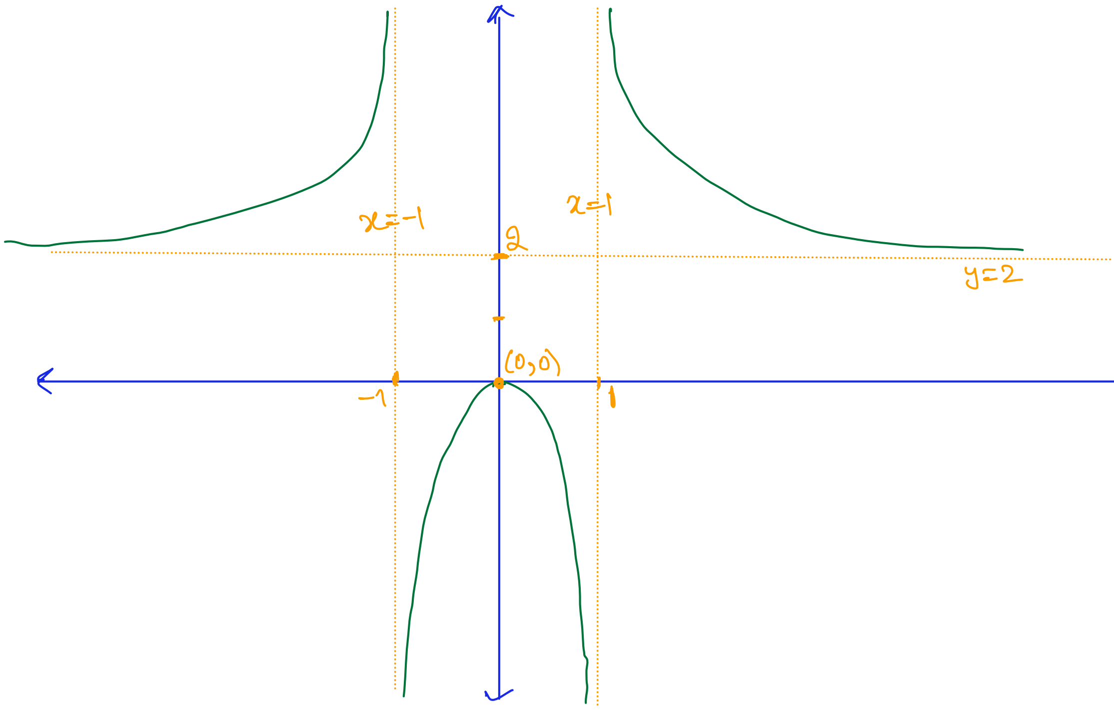
→ Passes through $(0,0)$ → even function.

→ Increasing in $(-\infty, 0)$ and decreasing in $(0, \infty)$

→ $x=1$, $x=-1$ and $y=2$ are asymptotes.

→ $(0,0)$ is L. max and no L. min.

→ Concave up in $(-\infty,-1) \cup (1,\infty)$ and concave down in $(-1,1)$



**Example 2**. Sketch the curve $y = \dfrac{x^2}{\sqrt{x+1}}$.

Domain : $x+1>0 \Rightarrow x>-1 \Rightarrow (-1, \infty)$

Intercepts : $f(0) = \dfrac{0^2}{\sqrt{0+1}} = 0 \Rightarrow (0,0)$ is a y-int and/or x-int.

$$\frac{x^2}{\sqrt{x+1}} = 0 \Rightarrow x^2 = 0 \Rightarrow x = 0$$

Symmetry : $f(-x) = \dfrac{(-x)^2}{\sqrt{-x+1}} = \dfrac{x^2}{\sqrt{-x+1}} \begin{matrix} \neq f(x) \\ \neq -f(x) \end{matrix} \Big\}$ neither even nor odd.

Asymptotes : $x=-1$ is a vertical asymptote.

$$\lim_{x\to\infty} \frac{x^2}{\sqrt{x+1}} = \lim_{x\to\infty} \frac{\frac{1}{\sqrt{x}}\, x^2}{\frac{1}{\sqrt{x}}\sqrt{x+1}} = \lim_{x\to\infty} \frac{x^{3/2}}{\sqrt{1+\frac{1}{x}}} = \infty$$

$\Rightarrow$ No horizontal asymptote.

Increase/Decrease/L.max/L.min

$$f'(x) = \frac{[x^2]'\sqrt{x+1} - x^2\left[\sqrt{x+1}\right]'}{(x+1)} = \frac{2x\sqrt{x+1} - x^2 \frac{1}{2\sqrt{x+1}}}{(x+1)}$$

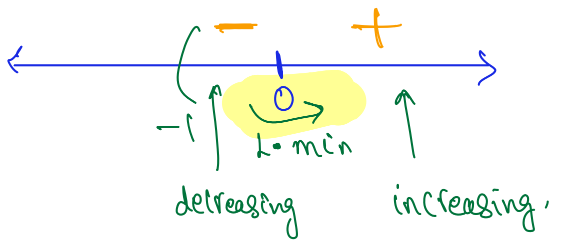
$$= \frac{\frac{4x(x+1) - x^2}{2\sqrt{x+1}}}{(x+1)} = \frac{4x^2+4x-x^2}{2(x+1)\sqrt{x+1}} = \frac{3x^2+4x}{2(x+1)\sqrt{x+1}}$$

$$= \frac{x(3x+4)}{2(x+1)\sqrt{x+1}}$$

(*) Domain $= (-1, \infty)$

$x > -1$

Critical numbers $\Rightarrow x = 0,\ \underbrace{x = -\frac{4}{3},\ x = -1}_{\text{not in domain.}}$




$$f'\left(-\tfrac{1}{2}\right) = \frac{-\frac{1}{2}\left(\frac{1}{2}\right)}{2\left(\frac{1}{2}\right)\sqrt{\frac{1}{2}}} < 0$$

## Concavity

$$f''(x) = \frac{[x(3x+4)]'\, 2(x+1)^{3/2} - x(3x+4)\left[2(x+1)^{3/2}\right]'}{4(x+1)^3}$$

$$\Rightarrow f''(x) = \frac{(6x+4)\,2(x+1)^{3/2} - x(3x+4)\,2\left(\tfrac{3}{2}\right)(x+1)^{1/2}}{4(x+1)^3}$$

$$= \frac{(x+1)^{1/2}\left[(6x+4)2(x+1) - 3x(3x+4)\right]}{4(x+1)^3}$$

$$= \frac{(x+1)^{1/2}\left[12x^2+8x+6x+8-9x^2-12x\right]}{4(x+1)^3}$$

$$= \frac{\overbrace{(x+1)^{1/2}}^{+ve}\left[3x^2+2x+8\right]}{\underbrace{4(x+1)^3}_{+ve}}$$

$3x^2+2x+8=0$ $\qquad x = \dfrac{-2 \pm \sqrt{(2)^2-4(3)(8)}}{6}$ $\qquad 4-96=-92$

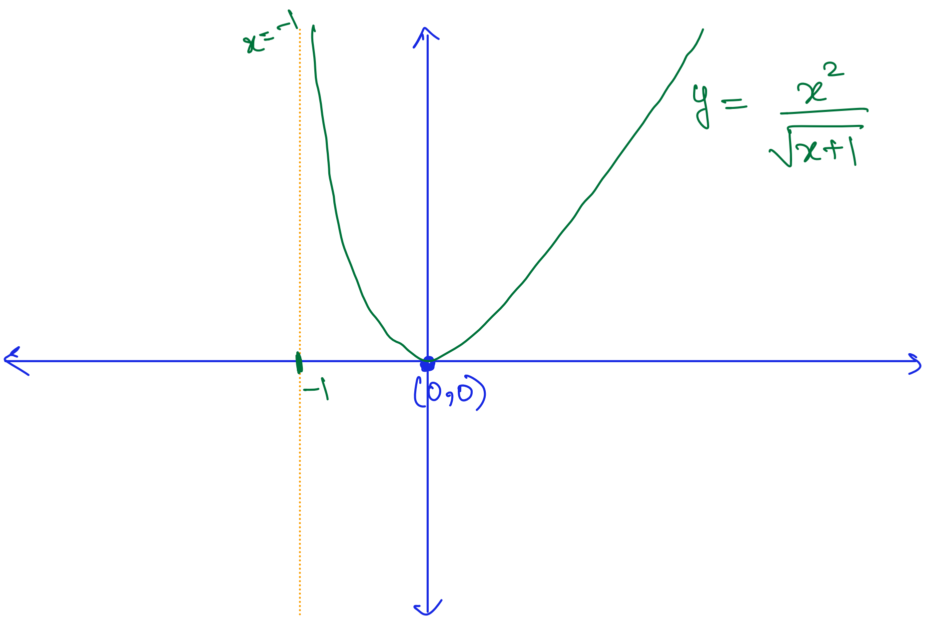
$\Downarrow$

no real roots.

$3x^2+2x+8>0$

$\Rightarrow f''(x) > 0$ for every $x>-1$

$\Rightarrow$ always Concave up.



**Example 3**. Sketch the curve $y = \dfrac{\cos x}{2 + \sin x}$.

Periodic : $y(x+2\pi) = \dfrac{\cos(x+2\pi)}{2+\sin(x+2\pi)} = \dfrac{\cos x}{2+\sin x} = y(x)$

Draw for $0 \le x \le 2\pi$ and then repeat the same curve to left and right.

Domain : $2+\sin x > 0$ since $-1 < \sin x < 1$.

Intercepts : $f(0) = \dfrac{\cos 0}{2+\sin(0)} = \dfrac{1}{2} \Rightarrow \left(0, \frac{1}{2}\right)$ is y-int.

$\Rightarrow \dfrac{\cos x}{2+\sin x} = 0 \Rightarrow \cos x = 0 \Rightarrow x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ (x-int)

Symmetry : $f(-x) = \dfrac{\cos(-x)}{2+\sin(-x)} = \dfrac{\cos x}{2-\sin x} \begin{matrix} \neq f(x) \\ \neq -f(x) \end{matrix} \Big\}$ neither even nor odd.

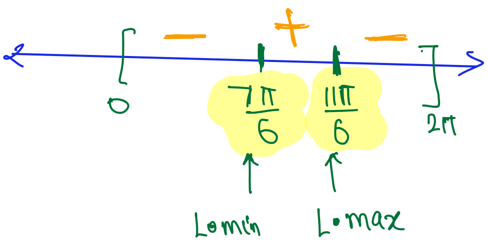
Asymptotes : No horizontal asymptote
No vertical asymptote

Increasing | Decreasing | L. max / min

$$f'(x) = \frac{(2+\sin x)[\cos x]' - \cos x[2+\sin x]'}{(2+\sin x)^2} = \frac{(2+\sin x)(-\sin x) - \cos x(\cos x)}{(2+\sin x)^2}$$

$$= \frac{-2\sin x - \sin^2 x - \cos^2 x}{(2+\sin x)^2} = \frac{-2\sin x - 1}{(2+\sin x)^2}$$

$-2\sin x - 1 = 0 \Rightarrow -2\sin x = 1 \Rightarrow \sin x = \dfrac{-1}{2}$




$\sin x < \dfrac{-1}{2}$ when $\dfrac{7\pi}{6} < x < \dfrac{11\pi}{6}$

$\Rightarrow 2\sin x + 1 < 0 \Rightarrow -2\sin x - 1 > 0$

$\sqrt{3} = 1.732$

$$f\left(\frac{7\pi}{6}\right) = \frac{\cos\frac{7\pi}{6}}{2+\sin\frac{7\pi}{6}} = \frac{-\cos\pi/6}{2-\sin\pi/6} = \frac{-\frac{\sqrt{3}}{2}}{2-\frac{1}{2}} = \frac{-\sqrt{3}/2}{3/2} = -\frac{\sqrt{3}}{3} = -0.57$$

$$f\left(\frac{11\pi}{6}\right) = \frac{\cos 11\pi/6}{2+\sin 11\pi/6} = \frac{\cos\pi/6}{2-\sin\pi/6} = \frac{\sqrt{3}/2}{2-\frac{1}{2}} = \frac{\sqrt{3}}{3} = 0.57$$

$$f''(x) = \frac{d}{dx}\left[\frac{-2\sin x - 1}{(2+\sin x)^2}\right]$$

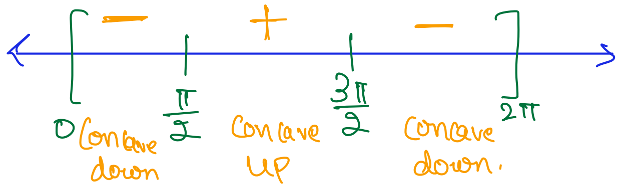
$$= \frac{(-2\sin x-1)'(2+\sin x)^2 - (-2\sin x-1)\left[(2+\sin x)^2\right]'}{(2+\sin x)^4}$$

$$= \frac{-2\cos x(2+\sin x)^2 + (2\sin x+1)\left[2(2+\sin x)\cos x\right]}{(2+\sin x)^4}$$

Factor $2(2+\sin x)\cos x$ in numerator.

$$= \frac{2\cos x(2+\sin x)\left[-(2+\sin x) + (2\sin x+1)\right]}{(2+\sin x)^4}$$

$$= \frac{2\cos x\left[-2-\sin x+2\sin x+1\right]}{(2+\sin x)^3} = \frac{2\cos x(\sin x-1)}{\underbrace{(2+\sin x)^3}_{\text{always +ve}}}$$




$f''(x)=0 \Rightarrow \cos x = 0$ or $\sin x = 1$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$ $\qquad$ $x = \pi/2$

$$f(x) = \frac{\cos x}{2+\sin x}$$