**Learning objectives**:

1. To apply the methods learned for finding extremal values of functions.
2. Solve problems such as maximizing or minimizing areas, distances, volumes, profit etc.

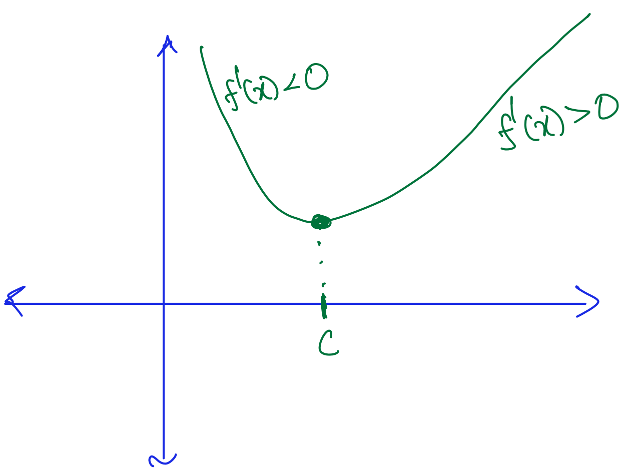
**Steps in solving optimization problems**

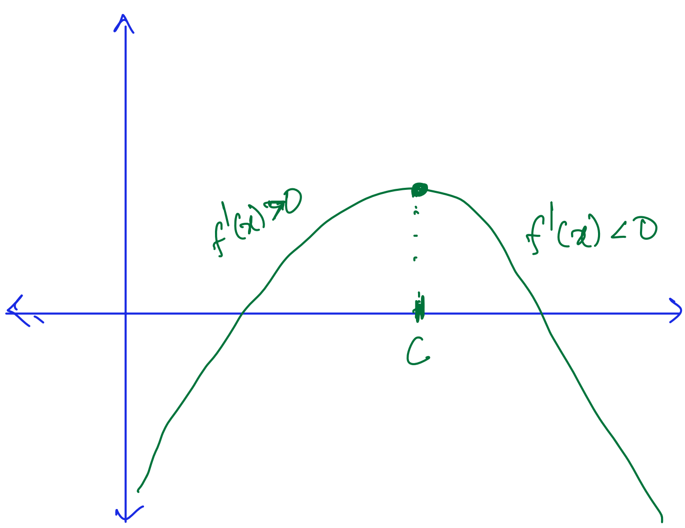
1. Understand the problem: Figure out which function needs to be minimized or maximized (**objective function**) and what are the **constraints**.
2. Draw a **diagram** if possible to correctly arrive at the objective function.
3. Introduce notation: The objective function depends on certain **variables**. Figure out what are those variables.
4. Express the objective function in terms of these variables.
5. Find **relationships** among the variables and the **constraints** on them.
6. Use methods of Sections 3.1 (closed interval method) and 3.3 (local maxima or minima) to find the optimal value of the objective function.

**First derivative test for absolute extremal values** (Only one critical pt contributes to sign change of $f'(x)$)

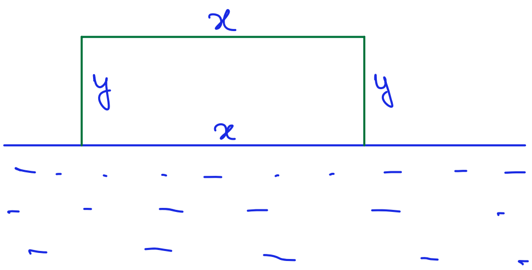
Suppose that $c$ is a critical number of a continuous function $f$ defined on an interval.

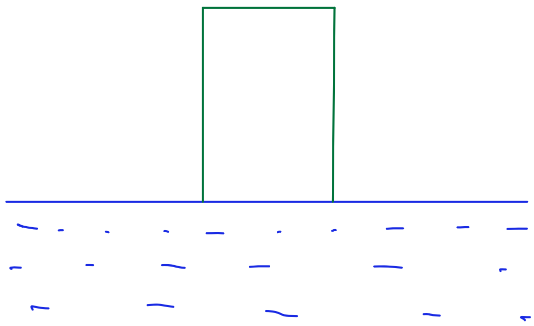
1. If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of $f$.
2. If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute ~~maximum~~ minimum value of $f$.



**Example 1**. Suppose you have 2400 ft of fencing and want to fence off a rectangular field that borders a straight river. You need no fence along the river. What are the dimensions of the field that has the largest area?



$$x + 2y = 2400 \quad , \quad A = xy$$

$$\Rightarrow 2y = 2400 - x$$

$$\Rightarrow y = \frac{1}{2}(2400 - x) = 1200 - \frac{1}{2}x$$

$$\Rightarrow A(x) = x\left(1200 - \frac{1}{2}x\right) = 1200x - \frac{1}{2}x^2$$

$$A'(x) = 1200 - x \quad \Rightarrow A'(x) = 0 \Rightarrow 1200 - x \Rightarrow \underbrace{x = 1200}_{\text{The only critical pt.}}$$

$$\xleftarrow{\quad + \quad} 1200 \xrightarrow{\quad - \quad}$$

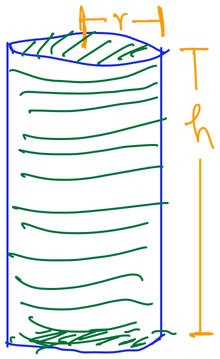
max. $\Rightarrow$ gives absolute max.

$$1200 + 2y = 2400 \Rightarrow y = 600$$

$\Rightarrow$ Dimensions maximizing area are $x = 1200$ ft, $y = 600$ ft

Maximum area $= (1200)(600) = 720000 \text{ ft}^2$

**Example 2**. A cylindrical can is to be made to hold 1 liter of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.




$$V = \pi r^2 h = 1$$

$$A = 2\pi r h + \pi r^2 + \pi r^2$$
$$= 2\pi r h + 2\pi r^2$$
$$= 2\pi r(h+r)$$

$$\Rightarrow \pi r^2 h = 1 \Rightarrow h = \frac{1}{\pi r^2}$$

$$\Rightarrow A(r) = 2\pi r\left(\frac{1}{\pi r^2} + r\right)$$

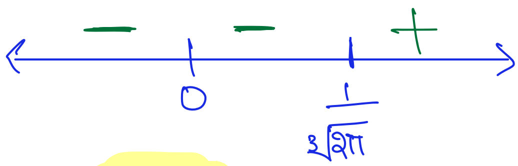
$$A'(r) = \frac{d}{dr}\left[\frac{2\pi r}{\pi r^2} + 2\pi r^2\right] = \frac{d}{dr}\left[\frac{2}{r} + 2\pi r^2\right]$$

$$= \frac{-2}{r^2} + 4\pi r \Rightarrow A'(r) = \frac{-2 + 4\pi r^3}{r^2}$$

$$\Rightarrow A'(r) = \frac{4\pi r^3 - 2}{r^2}$$ $\Rightarrow$ critical numbers: $4\pi r^3 - 2 = 0$, $r = 0$

$$\Rightarrow r = \left(\frac{2}{4\pi}\right)^{1/3} \text{ or } r = 0$$

$$\Rightarrow r = \frac{1}{\sqrt[3]{2\pi}} \text{ or } r = 0$$




$\Rightarrow r = \frac{1}{\sqrt[3]{2\pi}}$ minimizes the surface area,

$$h = \frac{1}{\pi r^2} = \frac{(2\pi)^{2/3}}{\pi}$$

$$= \frac{1}{\pi\left(\frac{1}{\sqrt[3]{2\pi}}\right)^2} = \frac{\left(\sqrt[3]{2\pi}\right)^2}{\pi}$$

## Alternatively

$$A = 2\pi rh + 2\pi r^2 \quad , \quad \pi r^2 h = 1$$

$$\frac{dA}{dr} = 2\pi \frac{d}{dr}(rh) + 4\pi r$$

$$\frac{dA}{dr} = 2\pi \left[\frac{d}{dr}(r)\right] h + 2\pi r \frac{d}{dr}(h) + 4\pi r$$

$$\overset{\parallel}{0} \quad = \boxed{2\pi h + 2\pi r \frac{dh}{dr} + 4\pi r = 0} \text{ ——— (I)}$$

Diff. $\pi r^2 h = 1$ w.r.t. $r$ :–

$$\pi \frac{d}{dr}(r^2 h) = \frac{d}{dr}(1)$$

$$\Rightarrow \pi h \frac{d}{dr}(r^2) + \pi r^2 \frac{dh}{dr} = 0$$

$$\Rightarrow \boxed{2\pi rh + \pi r^2 \frac{dh}{dr} = 0} \text{ ——— (II)}$$

Multiply (I) with $\frac{1}{2}r$ :–

$$\Rightarrow \frac{1}{2}r(2\pi h) + \frac{1}{2}r(2\pi r)\frac{dh}{dr} + \frac{1}{2}r(4\pi r) = 0$$

$$\Rightarrow \boxed{\pi rh + \pi r^2 \frac{dh}{dr} + 2\pi r^2 = 0} \text{ ——— (III)}$$

Subtract (II) from (III) :–

$$\pi rh + \cancel{\pi r^2 \frac{dh}{dr}} + 2\pi r^2 = 0$$

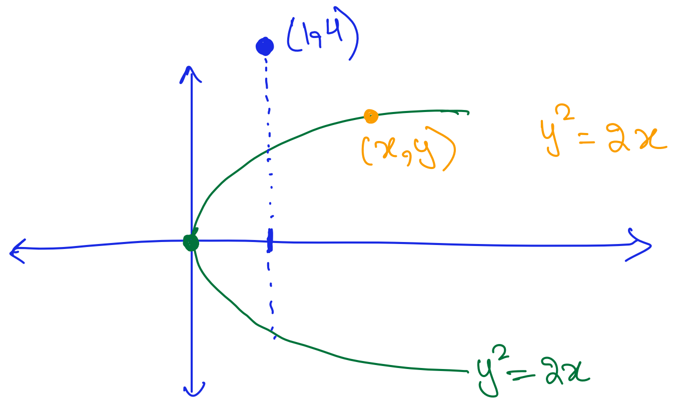
$$- 2\pi rh \ - \cancel{\pi r^2 \frac{dh}{dr}} \qquad = 0$$

$$-\pi rh + 2\pi r^2 = 0 \quad \Rightarrow 2\cancel{\pi}r^{\cancel{2}} = \cancel{\pi}\cancel{r}h$$

$$\Rightarrow \boxed{2r = h}$$

$$\left.\begin{array}{l} \pi r^2 h = 1 \\ h = 2r \end{array}\right\} \text{From both we get } r \text{ and } h.$$

**Example 3**. Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.




$$d(x,y) = \underbrace{\sqrt{(x-1)^2 + (y-4)^2}}_{\text{want to minimize this expression | function.}}$$

$$d(y) = \sqrt{\left(\frac{y^2}{2}-1\right)^2 + (y-4)^2}$$

Minimizing $d(y)$ is equivalent to minimizing $g(y) = \left(\frac{y^2}{2}-1\right)^2 + (y-4)^2$

want to minimize $\rightarrow g(y) = \left(\frac{y^2}{2}-1\right)^2 + (y-4)^2$

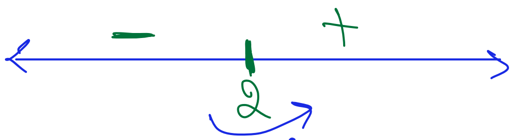
$$g'(y) = 2\left(\frac{y^2}{2}-1\right)\frac{d}{dy}\left(\frac{y^2}{2}-1\right) + 2(y-4)$$

$$= 2\left(\frac{y^2}{2}-1\right)(y) + 2(y-4) = (y^2-2)y + 2(y-4)$$

$$= y^3 - 2y + 2y - 8 = y^3 - 8$$

$\Rightarrow g'(y) = y^3 - 8$

$g'(y) = 0 \Rightarrow y^3 - 8 = 0 \Rightarrow y^3 = 8 \Rightarrow y = \sqrt[3]{8} \Rightarrow y = 2$

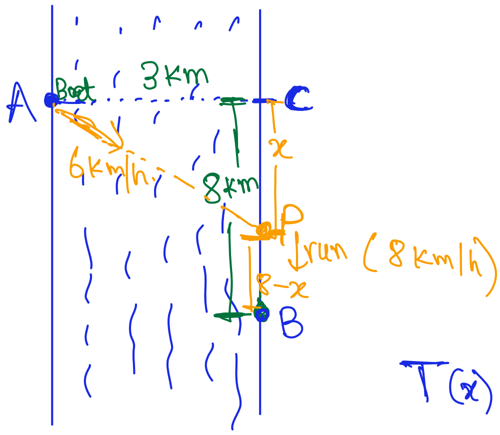



l. min $\Rightarrow$ absolute min because only one critical pt

$y = 2$ and $y^2 = 2x \Rightarrow 2^2 = 2x \Rightarrow x = 2$

$\Rightarrow$ $(2,2)$ is the point on $y^2 = 2x$ which is closest to $(1,4)$

**Example 4**. Suppose you launch your boat from point A on a bank of a straight river, 3 km wide, and want to reach point B, 8 km downstream on the opposite bank, as quickly as possible. You could row your boat directly across the river to point C and then run to B, or you could row directly to B, or you could row to some point D between C and B and then run to B. If you can row 6 km/h and run 8 km/h, where should you land to reach B as soon as possible? (We assume that the speed of the water is negligible compared with the speed at which you row.)




$PC = x$ km

$PB = 8-x$ km.

$AP = \sqrt{3^2 + x^2}$ km

$$T(x) = \frac{AP}{6} + \frac{PB}{8}$$

$$\Rightarrow T(x) = \frac{\sqrt{9+x^2}}{6} + \frac{8-x}{8} \quad \text{hours.}$$

$$T'(x) = \frac{1}{6}\left[\frac{1}{2\sqrt{9+x^2}} \times 2x\right] + \frac{1}{8}(-1)$$

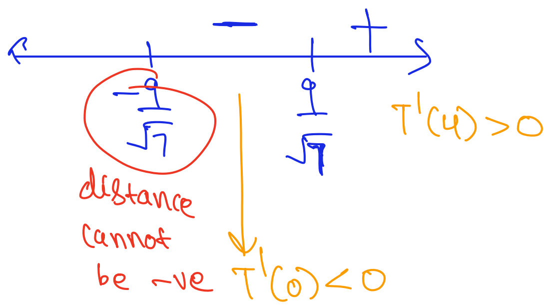
$$= \frac{x}{6\sqrt{9+x^2}} - \frac{1}{8} = 0$$

$$\Rightarrow \frac{x}{\sqrt{9+x^2}} = \frac{6}{8} = \frac{3}{4} \Rightarrow 4x = 3\sqrt{9+x^2}$$

Square both sides $\Rightarrow 16x^2 = 9(9+x^2)$

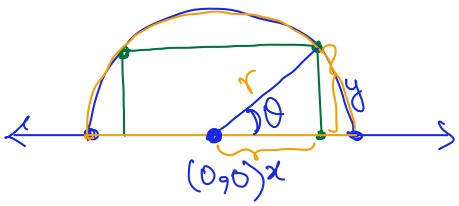
$$\Rightarrow 16x^2 = 81 + 9x^2 \Rightarrow 7x^2 = 81 \Rightarrow x^2 = \frac{81}{7}$$

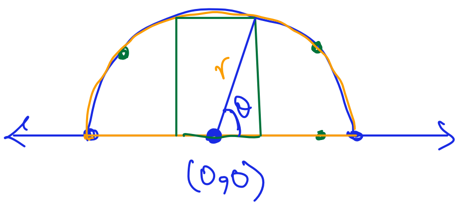
$$\Rightarrow x = \pm\frac{9}{\sqrt{7}}$$

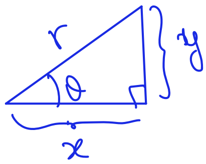



$$\Rightarrow x = \frac{9}{\sqrt{7}} = \frac{9\sqrt{7}}{7} \text{ km}$$

**Example 5**. Find the area of the largest rectangle that can be inscribed in a semicircle of radius $r$.




$$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$$

$$\Rightarrow A(\theta) = 2xy = 2(r\cos\theta)\,r\sin\theta$$

$$= 2r^2 \sin\theta\cos\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\Rightarrow A(\theta) = r^2 \underbrace{\sin 2\theta}_{\downarrow}$$

maximum when $2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$

maximum value of $\sin 2\theta = 1$

$\Rightarrow$ maximum area $= r^2$

or demand function

Let $p(x)$ be the price function of a quantity where $x$ is units of the quantity.

Then the revenue function is given by $R(x) = xp(x)$.

If the cost function is $C(x)$, then profit is given by $R(x) - C(x)$. $= x\,P(x) - C(x)$

**Example 6**. A store has been selling 200 flat-screen TVs a week at \$350 each. A market survey indicates that for each \$10 rebate offered to buyers, the number of TVs sold will increase by 20 a week. Find the demand function (assuming it is linear) and the revenue function. How large a rebate should the store offer to maximize its revenue?

$$P(x) = ax + b$$

$$P(200) = 350 \quad \text{and} \quad P(220) = 340$$

$$200a + b = 350 \quad \text{and} \quad 220a + \cancel{b} = 340$$

$$-\;200a\; - \cancel{b} = -350$$

$$20a = -10$$

$$\Rightarrow a = \frac{-1}{2}$$

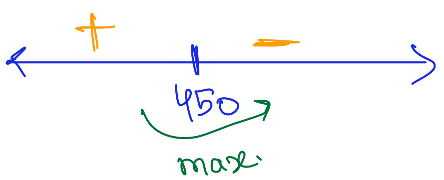
$$200\left(-\frac{1}{2}\right) + b = 350 \Rightarrow -100 + b = 350$$

$$\Rightarrow b = 450 \Rightarrow P(x) = 450 - \frac{1}{2}x$$

$$R(x) = x\,P(x) = x\left(450 - \frac{1}{2}x\right) = 450x - \frac{1}{2}x^2$$

$$R'(x) = 450 - \frac{1}{2}(2x) = 450 - x$$

$$R'(x) = 0 \Rightarrow 450 - x = 0 \Rightarrow x = 450$$




$\Rightarrow x = 450$ corresponds to an absolute max revenue

$$P(450) = 450 - \frac{1}{2}(450) = 225\,\$ \Rightarrow \text{Rebate} = (350 - 225)\,\$ = 125\,\$$$