Learning objectives:

1. What are antiderivatives?
2. How to find antiderivatives of functions?
3. Applications to straight line motion.

Antiderivative

A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

Theorem

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + c$$

where c is an arbitrary constant.

Example 1. Find the most general antiderivatives of the following functions.

1. $f(x) = \sin x$.
2. $f(x) = x^2$.
3. $f(x) = x^{-3}$.

(2) $f(x) = x^2 \xrightarrow{\text{Antiderivative}} F(x) = \frac{1}{3}x^3 + C$

(3) $f(x) = x^{-3} \xrightarrow{\text{Antiderivative}} F(x) = \frac{1}{-3+1}x^{-3+1} + C$

$$= \frac{-1}{2}x^{-2} + C$$

(1) $\frac{d}{dx}(-\cos x) = -(-\sin x) = \sin x$

$f(x) = \sin x \xrightarrow{\text{Antiderivative}} F(x) = -\cos x + C$

Antiderivatives of sums and constant multiples

1. If F is an antiderivative of f then $cF(x)$ is an antiderivative of $cf(x)$.
2. If F and G are antiderivative of f and g respectively then an antiderivative of $f(x) + g(x)$ is $F(x) + G(x)$.

Antiderivatives of common functions

Function	Most general antiderivative
x^n , $n \neq -1$	$\frac{x^{n+1}}{n+1} + c$
$\cos x$	$\sin x + c$
$\sin x$	$-\cos x + c$
$\sec^2 x$	$\tan x + c$
$\sec x \tan x$	$\sec x + c$
$\csc^2 x$	$-\cot x + c$
$\csc x \cot x$	$-\csc x + c$

Example 2. Find the most general antiderivative of $g(x) = 4 \sin x + \dfrac{2x^5 - \sqrt{x}}{x}$.

$$G(x) = 4 \underbrace{\text{Antiderivative}}_{\text{Ad}}(\sin x) + \underbrace{\text{Antiderivative}}_{\text{Ad}}\left(\frac{2x^5 - \sqrt{x}}{x}\right) + C$$

$$\text{Ad}(\sin x) = -\cos x$$

$$\text{Ad}\left(\frac{2x^5 - \sqrt{x}}{x}\right) = \text{Ad}\left(\frac{2x^5}{x} - \frac{\sqrt{x}}{x}\right)$$

$$= \text{Ad}\left(2x^{5-1} - x^{\frac{1}{2}-1}\right)$$

$$= \text{Ad}\left(2x^4 - x^{-\frac{1}{2}}\right) = 2\,\text{Ad}(x^4) - \text{Ad}(x^{-\frac{1}{2}})$$

$$= 2\frac{x^{4+1}}{4+1} - \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = 2\frac{x^5}{5} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}}$$

$$G(x) = 4(-\cos x) + \frac{2}{5}x^5 - 2x^{\frac{1}{2}} + c = -4\cos x + \frac{2}{5}x^5 - 2\sqrt{x} + C$$

Example 3. Find f if $f'(x) = x\sqrt{x}$ and $\underbrace{f(1) = 2}_{\text{initial condition}}$.

f is an antiderivative of f'

$\Rightarrow f(x) = Ad(x\sqrt{x}) + C$

$$Ad(x \cdot x^{1/2}) = Ad(x^{1+\frac{1}{2}}) = Ad(x^{3/2}) = \frac{x^{3/2+1}}{\frac{3}{2}+1}$$

$$= \frac{x^{5/2}}{5/2} = \frac{2}{5}x^{5/2}$$

$f(x) = \frac{2}{5}x^{5/2} + C \quad \Rightarrow f(1) = 2 \Rightarrow \frac{2}{5}(1)^{5/2} + C = 2$

$\Rightarrow C = 2 - \frac{2}{5} = \frac{8}{5}$

$\Rightarrow f(x) = \frac{2}{5}x^{5/2} + \frac{8}{5}$

Example 4. Find f if $f''(x) = 12x^2 + 6x - 4$, $\underbrace{f(0) = 4 \text{ and } f(1) = 1}_{\text{two initial conditions.}}$.

$f'(x) = Ad(f''(x))$

$= Ad(12x^2 + 6x - 4) + C = 12\, Ad(x^2) + 6\, Ad(x) - 4\, Ad(x^0) + C$

$= 12\left(\frac{x^{2+1}}{2+1}\right) + 6\left(\frac{x^{1+1}}{1+1}\right) - 4\left(\frac{x^{0+1}}{0+1}\right) + C$

$\Rightarrow f'(x) = 4x^3 + 3x^2 - 4x + C$

$f(x) = Ad(f'(x)) = 4\, Ad(x^3) + 3\, Ad(x^2) - 4\, Ad(x) + C\, Ad(x^0) + d$

$= 4\frac{x^{3+1}}{3+1} + 3\frac{x^{2+1}}{2+1} - 4\frac{x^{1+1}}{1+1} + C\frac{x^{0+1}}{0+1} + d$

$= x^4 + x^3 - 2x^2 + Cx + d$

$f(0) = 4 \quad \Rightarrow 0^4 + 0^3 - 2(0)^2 + C(0) + d = 4 \Rightarrow d = 4$

$f(1) = 1 \quad \Rightarrow (1)^4 + (1)^3 - 2(1)^2 + C(1) + d = 1 \Rightarrow 1 + 1 - 2 + C + 4 = 1$

$\Rightarrow C + 4 = 1 \Rightarrow C = -3$

$\Rightarrow f(x) = x^4 + x^3 - 2x^2 - 3x + 4$

Example 5. A particle moves in a straight line and has acceleration given by $a(t) = (6t + 4)$ cm/s^2. Its initial velocity is $v(0) = -6$ cm/s and its initial displacement is $s(0) = 9$ cm. Find its position function $s(t)$.

$v(t) = s'(t)$ and $a(t) = v'(t)$

$\Rightarrow v(t) = Ad(a(t)) = Ad(6t+4) + c$

$= 6\, Ad(t) + 4\, Ad(t^0) + c = 6\frac{t^2}{2} + 4t + c$

$\Rightarrow v(t) = 3t^2 + 4t + c$ and $v(0) = -6$ cm/s

$\Rightarrow -6 = 3(0)^2 + 4(0) + c \Rightarrow c = -6$

$\Rightarrow v(t) = (3t^2 + 4t - 6)$ cm/s.

$\Rightarrow s(t) = Ad(v(t)) = 3\, Ad(t^2) + 4\, Ad(t) - 6\, Ad(t^0) + d$

$= 3\frac{t^3}{3} + 4\frac{t^2}{2} - 6t + d = t^3 + 2t^2 - 6t + d$

$\Rightarrow s(0) = 9 \Rightarrow d = 9$ $\Rightarrow s(t) = t^3 + 2t^2 - 6t + 9$

Example 6. A ball is thrown upward with a speed of 48 ft/s from the edge of a cliff 432 ft above the ground. Find its height above the ground t second later. When does it reach its maximum height? When does it hit the ground? Use the value of acceleration due to gravity to be -32 ft/s^2.

$h(t) \rightarrow$ height function. $\Rightarrow v(t) = h'(t)$, $a(t) = v'(t)$

$a(t) = -32 \Rightarrow v(t) = Ad(-32) + c = -32\, Ad(t^0) + c$

$\Rightarrow v(t) = -32\frac{t^{0+1}}{0+1} + c = -32t + c$

$\Rightarrow v(0) = 48$ ft/s $\Rightarrow c = 48 \Rightarrow v(t) = -32t + 48$

$h(t) = Ad(v(t)) = -32\, Ad(t) + 48\, Ad(t^0) + d$

$= -32\frac{t^2}{2} + 48t + d = -16t^2 + 48t + d$

$h(0) = 432$ ft $\Rightarrow d = 432$

$\Rightarrow h(t) = -16t^2 + 48t + 432$

$v(t) = 0$ (solve for t)

$\Rightarrow -32t + 48 = 0 \Rightarrow 32t = 48 \Rightarrow t = \frac{48}{32} = \frac{3}{2}$ s.

$h\left(\frac{3}{2}\right)$ = maximum height.

$h(t) = 0$ (solve for t)

$\Rightarrow -16t^2 + 48t + 432 = 0$

$\Rightarrow -16(t^2 - 3t - 27) = 0 \Rightarrow t^2 - 3t - 27 = 0$

$$t = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-27)}}{2(1)}$$

$$\Rightarrow t = \frac{3 \pm \sqrt{9 + 108}}{2} = \frac{3 \pm \sqrt{117}}{2}$$

$\Rightarrow t$ cannot be $\frac{3 - \sqrt{117}}{2}$ since its negative.

$\Rightarrow t = \frac{3 + \sqrt{117}}{2}$ s. $\rightarrow$ hits the ground

$117 = 13 \times 9$

$= \frac{3 + 3\sqrt{13}}{2}$ s.