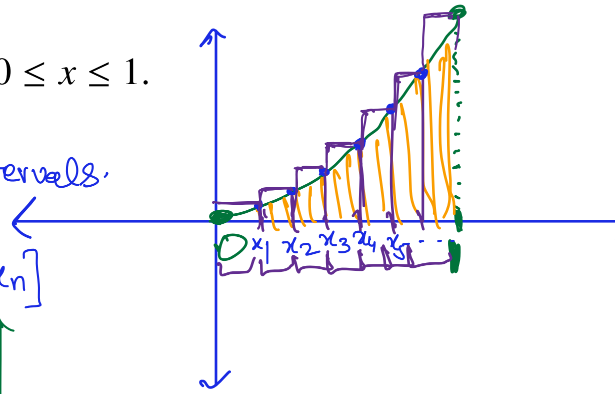
Learning objectives:

1. Express areas under curves as limit of a sum.
2. Apply this to calculating distance.



Example 1. Find the area under the curve $y = x^2$ for $0 \le x \le 1$.

• Divide $[0, 1]$ into n number of subintervals.

$[x_0, x_1]$, $[x_1, x_2]$, $[x_2, x_3]$, ..., $[x_{n-1}, x_n]$

1^{st} subinterval, 2^{nd}, 3^{rd}, ... n^{th} subinterval

• We draw the vertical lines $x = x_i$, $i = 0, 1, 2, 3, \ldots, n$

$x_0 = 0$, $x_n = 1$

• Draw a rectangle having width $= x_i - x_{i-1}$
and height $= f(x_i)$

• Find area of every rectangle and sum all the areas.

i^{th} rectangle has area $= f(x_i)(x_i - x_{i-1})$

Sum $= f(x_1)(x_1 - x_0) + f(x_2)(x_2 - x_1) + \ldots\ldots + f(x_n)(x_n - x_{n-1})$

• Assume every subinterval has same width. Δx

$\Rightarrow x_i - x_{i-1} = \Delta x$ for $i = 1, 2, \ldots, n$

n subinterval each of width $\Delta x \Rightarrow n(\Delta x) =$ length of $[0,1]$ $= 1$

$\Delta x = \frac{1}{n}$

$x_0 = 0$, $x_1 - x_0 = \Delta x = \frac{1}{n} \Rightarrow x_1 = x_0 + \frac{1}{n} = \frac{1}{n}$

$x_2 - x_1 = \Delta x = \frac{1}{n} \Rightarrow x_2 = x_1 + \frac{1}{n} = \frac{2}{n}$

$x_3 - x_2 = \Delta x = \frac{1}{n} \Rightarrow x_3 = x_2 + \frac{1}{n} = \frac{3}{n}$

$\vdots$

At i^{th} step, $x_i = \frac{i}{n}$

$\vdots$

when $i=n$, $x_n = \frac{n}{n} = 1$

$$\text{Sum} = f(x_1)(x_1-x_0) + f(x_2)(x_2-x_1) + \cdots\cdots + f(x_n)(x_n - x_{n-1})$$

$$= f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$$

$$= \frac{1}{n} f(x_1) + \frac{1}{n} f(x_2) + \cdots + \frac{1}{n} f(x_n)$$

$$= \frac{1}{n}\left[f(x_1) + f(x_2) + \cdots + f(x_n)\right]$$

$$= \frac{1}{n}\left[f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + f\left(\frac{3}{n}\right) + \cdots + f\left(\frac{i}{n}\right) + \cdots + f\left(\frac{n}{n}\right)\right]$$

$$= \frac{1}{n}\left[\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \left(\frac{3}{n}\right)^2 + \cdots + \left(\frac{i}{n}\right)^2 + \cdots + \left(\frac{n}{n}\right)^2\right]$$

$$= \frac{1}{n}\left[\frac{1^2}{n^2} + \frac{2^2}{n^2} + \frac{3^2}{n^2} + \cdots + \frac{i^2}{n^2} + \cdots + \frac{n^2}{n^2}\right]$$

$$= \frac{1}{n^3}\left[1^2 + 2^2 + 3^2 + \cdots + i^2 + \cdots + n^2\right]$$

$$= \frac{1}{n^{\cancel{3}\,2}} \frac{\cancel{n}(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2}$$

$$A = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2}$$

$$= \lim_{n\to\infty} \frac{1}{6}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) = \lim_{n\to\infty} \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

$$= \frac{1}{6}(1+0)(2+0) = \frac{1}{3}$$

If we had $[a, b]$, then $n(\Delta x) = b-a \Rightarrow \Delta x = \frac{b-a}{n}$

$x_0 = a$, $x_1 = a + \Delta x$, $x_2 = a + 2(\Delta x)$, ……, $x_i = a + i(\Delta x)$

$x_n = a + n(\Delta x) = a + b - a = b$

$x_i = a + \frac{i(b-a)}{n}$, $i = 0, 1, 2, \ldots\ldots, n$

Area as limit of a sum

The area A of the region S that lies under the graph of a continuous function f is the limit of the sum of the areas of approximating rectangles.

sum of areas of rectangles

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \left(f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x\right) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x\,.$$

$$x_i = a + \frac{i(b-a)}{n}, \quad \Delta x = \frac{b-a}{n}$$

Σ- notation for a sum.

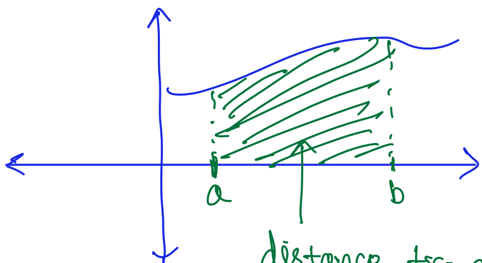
Distance

Distance is the area under the graph of the velocity function.

Example 2. An object starts to move at $t = 0$ with a velocity that varies with time as $v(t) = t^3$. Find the distance covered up to time $t = 4$ seconds.

$$v(t) = \frac{ds}{dt} \simeq \frac{\Delta s}{\Delta t} \Rightarrow \Delta s = v(t)\,\Delta t$$

Distance = area under graph of $v(t) = t^3$ in the interval $[0, 4]$



distance travelled starting $t=a$ upto $t=b$

Step 1

$a = 0, \; b = 4$

$$\Delta x = \frac{4-0}{n} = \frac{4}{n}$$

$$x_i = a + i(\Delta x) = 0 + i\left(\frac{4}{n}\right) \Rightarrow x_i = \frac{4i}{n}$$

Step 2

$$R_n = v(x_1)\,\Delta x + v(x_2)\,\Delta x + \cdots\cdots + v(x_n)\,\Delta x$$

$$= \frac{4}{n}\left[x_1^3 + x_2^3 + \cdots\cdots + x_n^3\right]$$

$$= \frac{4}{n}\left[\left(\frac{4}{n}\right)^3 + \left(\frac{4\cdot 2}{n}\right)^3 + \left(\frac{4\cdot 3}{n}\right)^3 + \cdots\cdots + \left(\frac{4\cdot n}{n}\right)^3\right]$$

$$= \frac{4}{n}\left[\left(\frac{4}{n}\right)^3\cdot 1^3 + \left(\frac{4}{n}\right)^3\cdot 2^3 + \left(\frac{4}{n}\right)^3\cdot 3^3 + \cdots\cdots + \left(\frac{4}{n}\right)^3 n^3\right]$$

$$= \frac{4}{n}\cdot\left(\frac{4}{n}\right)^3\left[1^3 + 2^3 + 3^3 + \cdots\cdots + n^3\right]$$

$$= \left(\frac{4}{n}\right)^4 \frac{n^2(n+1)^2}{4} = \frac{\cancel{4^4}^{\,3}}{\cancel{n^4}_{\,2}}\,\frac{\cancel{n^2}(n+1)^2}{\cancel{4}} = \frac{64(n+1)^2}{n^2}$$

Step 3

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{64(n+1)^2}{n^2}$$

$$= 64 \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 = 64\left(\lim_{n\to\infty} \frac{n+1}{n}\right)^2$$

$$= 64\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\right)^2$$

$$= 64(1+0)^2 = \boxed{64}$$

distance travelled in the first four seconds